Some notes on calculating particle orbits.

Our goal is to track proton orbits in the Taylor state fields for 10^6 particles for 1000's (or millions) of orbits. The protons should be drawn from a distribution $F(v_\parallel, v_\perp)$. Effectively what that means is we want to know about the dynamics of protons of different energies, and different pitch angles. Particles with low pitch angle (ie low $v_\perp/v_\parallel$) will stream along field lines, but particles with high pitch angle (ie high $v_\perp/v_\parallel$) might get stuck between regions of high field (see my mirror notes). Some high energy protons will have large orbits and will hit the wall. Others might drift out. We'll want a plot of $log(N(t))$,

From earlier notes, here's the Lorentz force equation for charged particle orbits in arbitrary electric and magnetic fields, with the charge-to-mass dependence explicitly pulled out:

$$m\frac{d\mathbf{v}}{dt} = \frac{Z}{M}q\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]$$

Note that m and q are really just fixed at unity and could be dropped. The dynamics of different ions (or even electrons) is in the Z/M term. A proton has Z/M=1, and we will start with no electric field so we have:

$$\frac{d\mathbf{v}}{dt} = \left[\mathbf{v}\times\mathbf{B}\right]$$

We pick a 1 eV proton in a 0.1 T field (SSX case), then an initial value of $v = 1$ corresponds to units of about $10^4\ m/s$. A time of $t = 1$ corresponds to a proton orbit time in a 0.1 T field (about 1 μs). Notice that this simulation would also apply for a 1 Tesla field, except the time step would proportionally shorter since the gyro-frequency would be 10 times higher. The orbit would be tighter and the acceleration dv/dt would be higher. The proton velocity would be the same (a 1 eV proton has the same velocity regardless).

1. Test case with uniform field, B_0**.** The idea here is to set $v = 1$ and $B_0 = 1$, then the Larmor radius:

$$r = \frac{mv}{qB} = 1$$

We should test this and verify. First, is it a circle? We need some kind of metric to see how close the circumference is to 2π (or area is to π). Second, is the radius constant? A plot of r as a function of iterations should be a straight line at $r = 1$, but because of accumulating error, the radius will drift. We'd like r to be accurate to within, say, 1% after, say, 1000 iterations (just a guess). The orbit integrator could be a leap-frog scheme, or a Runga-Kutta (4th order is good). Let's see what's the most accurate. Of course, if it takes a week to compute the orbit, that's no good, so there's a trade-off between accuracy and expedience. We'll want to do 10^6 protons in a few hours or so.

2. Other analytical fields. Like we discussed, we can test our orbit integrator on other analytical fields (there aren't many). One is a straight, current-carrying wire. Here the field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{B_0 R}{r}.$$

The drifts are well-known. For a proton with an initial velocity in the direction of the current, the particle will hop along the wire. Another is a dipole field, in which case the proton should bounce between north and south poles and gradually drift around the equator.

A little harder would be to calculate the orbits in an axisymmetric, m=0 Taylor state (ie a spheromak). The field components (B_r, B_ϕ, B_z) are written in both the Finn (eq. 8) or Bondeson (eq. 6) paper (I like Bondeson slightly better). One has to be careful to rationalize the intrinsic scale of the orbit $r = \frac{mv}{qB}$ with the intrinsic scale of the equilibrium given by $\nabla \times B = \lambda B$. In other words, for a spheromak in a can of radius a and length L, you can't just set a and L to unity. They will be a certain number of Larmor radii. If $L = a$, then that's a nice tuna can shape but L might be 20 or 100 or something. I won't elaborate further, but you'll also have a $k_r = 3.8317/a$ and a $k_z = \pi/L$ floating around. Also $\lambda_0^2 = k_r^2 + k_z^2$.

One can also do proton orbits in a Solovév geometry. See the Bellan book (and his paper) for details. The Solovév geometry is a good model for a tokamak. Again, there's an intrinsic scale in the Solovév geometry that has to be rationalized with the Larmor orbit scale. There is also a parameter α which determines the elongation of the geometry.

3. Taylor state fields. Finally, we want to see where protons go if let them loose in a Taylor state equilibrium. Ultimately, we will want to track orbits in dynamical fields. That will be **much** harder since we will have new magnetic fields at each time step, **plus** there will be electric fields from $\nabla \times E = -dB/dt$. For any dynamical simulation, there is also the interpolation problem discussed below. For now, we want to just see if a relaxed Taylor state is a good proton trap.

A good start will be the PSI-TET generated fields of a 10:1 Taylor state. This is the aspect ratio of our original configuration (see Gray, et al). We also studied a smaller aspect ratio (see Cothran, et al), and our present configuration has an aspect ratio of 20:1. Same issue applies here as above, the Taylor state has an intrinsic scale that is inconsistent with the proton gryo scale. A thermal proton has a gyro radius of unity in our scheme, so depending on the T_p and B_0 we choose, the radius of the flux conserver might be 20 units, and the length 200 units.

The new problem here is that there isn't a simple equation for the magnetic field components everywhere in space. At each time step, the proton will want to know the local field. Chris Hanson (UW) has sampled the field on a cylindrical grid with 20 radial, 72 angular, and 200 axial points. This is 288,000 points, which seems like a lot, but in the SSX case, L is about a meter, so the axial resolution is 5 mm. A 1 eV proton will execute several orbits in that distance, so we need to be able to interpolate fields between grid points.

There are lots of options here. Simplest is a linear interpolation, but this will give rise to unphysical kinks at the grid points ($\nabla \times B$ and $\nabla \cdot B$ are both zero here, so the fields are very smooth). Another option is some kind of spline or polynomial interpolation (Chebyshev polynomials have good properties). Finally, the way PSI-TET works is it solves the Galerkin eigenvalue problem for the force-free Taylor state: $\nabla \times B = \lambda B$. The way it does it is to expand the fields in some polynomial basis (see papers by Morse, Hua, Chandrasekhar/Kendall, Dandurand), so the PSI-TET solution has all those polynomial links between grid points. This data set is so large that it makes more sense to do particle orbits within the PSI-TET framework somehow.