Kolmogorov scaling (different indices)

The essence of the Kolmogorov 1941 scaling argument for the omni-directional wavenumber spectrum for fully developed turbulence is that $E(k)$ depends only on k (via a power-law) and also on the energy transfer rate ϵ. Kolmogorov thought about an energy rate per unit mass: $\epsilon \sim v^2/\tau$. For us, we think about magnetic energy so $\epsilon \sim b^2/\tau$, where b is the fluctuating part of the magnetic field, and τ is the time scale over which the energy is transferred.

The dimensions of $E(k)$ are such that:

$$\int E(k)dk = \langle b^2 \rangle$$

so $E(k) \propto b^2/k$. The time τ in the energy transfer rate depends on the physics of the transfer. For MHD, we consider an Alfvén crossing time at the scale L:

$$\tau_{MHD} = \frac{L}{v_A} \sim \frac{1}{kb}.$$

This is because $\omega_{MHD} = kv_A$. So now we do dimensional analysis:

$$E(k, \epsilon) = Ck^\alpha \epsilon^\beta$$

$$\frac{b^2}{k} = Ck^\alpha \left(\frac{b^2}{\tau_{MHD}} \right)^\beta = Ck^\alpha b^{2\beta} (kb)^\beta$$

We find that $2 = 3\beta$ or $\beta = 2/3$ and $-1 = \alpha + \beta$ so $\alpha = -5/3$. We get the famous Kolmogorov 1941 result:

$$E(k) = Ck^{-5/3} \epsilon^{2/3}$$

An interesting twist happens if the time scale for the transfer is faster, say due to Whistler waves or kinetic Alfvén waves. In that case, there's a different dispersion relation (see below). We get that $\omega_{Hall} = k^2 \delta_i v_A = k^2 \delta_e^2 \omega_{ce}$, or essentially:

$$\tau_{Hall} \sim \frac{1}{k^2 b}.$$

That extra factor of k changes the scaling for $E(k)$ at scales smaller than δ_i.

$$E(k, \epsilon) = Ck^\alpha \epsilon^\beta$$

$$\frac{b^2}{k} = Ck^\alpha \left(\frac{b^2}{\tau_{Hall}}\right)^\beta = Ck^\alpha b^{2\beta}(k^2 b)^\beta$$

We find that $2 = 3\beta$ or $\beta = 2/3$ and $-1 = \alpha + 2\beta$ so $\alpha = -7/3$. We get a modified energy spectrum:

$$E_{Hall}(k) = Ck^{-7/3}\epsilon^{2/3}$$

Dispersion relations: The dispersion relation for Whistler waves comes from the dispersion relation for R-waves (see Bellan, or any plasma book):

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2/\omega^2}{1 - \omega_{ce}/\omega}.$$

For SSX, the frequencies are always low compared to electron physics so $\omega \ll \omega_{pe}, \omega_{ce}$, so

$$\frac{c^2k^2}{\omega^2} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}\omega}.$$

Furthermore, SSX plasmas are always over-dense, meaning $\omega_{pe}/\omega_{ce} \gg 1$ (about 100 typically). So the dispersion relation becomes:

$$\frac{c^2k^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega_{ce}\omega}$$

$$\omega_{Hall} = \frac{c^2k^2\omega_{ce}}{\omega_{pe}^2} = \delta_e^2\omega_{ce}k^2 = \delta_i v_A k^2.$$

The key point is that the dispersion relation depends on k^2 (i.e. is dispersive) but it turns out that $\delta_e^2\omega_{ce} = \delta_i v_A$ (which is also interesting).

On my website, there are some notes called alpha scaling, but the basic story is that:

$$\alpha \equiv \tau_{Alf}\omega_{ci} = L\omega_{ci}/v_A = L/\delta_i$$

this says the number of orbits an ion executes in a characteristic dynamical time (the time it takes an Alfvénic disturbance to move a distance L) is the same as the number ion inertial lengths in L. Another way to write it is $v_A = \delta_i\omega_{ci}$. From there, it's easy to show that $\delta_i v_A = \delta_e^2\omega_{ce}$ (i.e. the form I used in the dispersion relation above) by keeping track of factors of M/m.