Physics 138
Plasma Physics 2003
Seminar 4/5 notes (MHD)

General:
Seminar 4: I think we had a good introduction into the derivation and structure of the single fluid MHD equations. I hope we have a good sense of the applicability of MHD: $\rho_i/L \ll 1, R_m \gg 1, \lambda_D/L \ll 1, v_{Alf}/c \ll 1$ which is the same as Goldston’s large dielectric constant argument $1 + \rho/\epsilon_0 B^2 \gg 1$. We also had a good introduction into MHD equilibria (the Bennett pinch and the simple solution to the Grad-Shafranov equation).

Seminar 5: Some excellent work on details of the Solov’ev equilibrium and MHD/fluid turbulence! I really liked the IDL particle codes (some posted on the website) and the Matt/Chris discussion of turbulence was excellent. It’s remarkable that Matt code a Poiseuille fluid code to run!

\[ \mathbf{B} = \nabla \psi \times \nabla \xi \] note: We discussed at length the construction $\mathbf{B} = \nabla \psi \times \nabla \xi$. We know that the div of both sides is zero but if we “undo” the div, we have the curl of an arbitrary vector function left over (as a constant of integration): $\mathbf{B} = \nabla \psi \times \nabla \xi + \nabla \times G$. It certainly makes sense that two arbitrary scalars $\psi, \xi$ should be enough to describe a divergence-less vector $\mathbf{B}$ but we couldn’t prove it mathematically.

\[ \nabla I \parallel \nabla \psi: \] We were unable to mathematically prove that if $\nabla I$ is parallel to $\nabla \psi$ then $I = I(\psi)$ but the argument is plausible. Chris points out that if you consider an isosurface of $I$, $\nabla I$ is everywhere normal to it. If $\nabla \psi$ is everywhere parallel to $\nabla I$, then the isosurface of $I$ is also an isosurface of $\psi$.

ignorable coordinates in $\psi$: Some confusion about the the separation of variables in eq. 8.3. The Bessel equation (8.4, 5) results from considering the $r$ dependence. Bellan says that “this configuration is axially and azimuthally uniform so that both $\theta$ and $z$ are ignorable coordinates” then the next sentence says, “Fourier analysis of eq. 8.3 implies that $\psi$ can be expressed as a linear superposition of modes varying as $exp(i m \theta + i k z)$.” If $\theta$ and $z$ are ignorable, then $m$ and $k$ are zero. Maybe we want to write $\psi \rightarrow \psi Z \Theta$ then show that the harmonic nature of Laplace’s equation gives $exp(i m \theta + i k z)$.

Solov’ev solution: How do we solve the diffeq 8.54 to get the Solov’ev solution 8.55? We discussed this and convinced ourselves that a series solution of 8.54 would yield 8.55 (we didn’t work it out though).

Pinch relation: Bellan’s pinch relation eq. 8.29, is inconsistent with Goldston’s by a factor of 2 (he has $4\pi^2$ not $8\pi^2$). In eq. 8.27, notice that
the $8\pi^2$ comes from the two factors of $2\pi$ in $J \times B$ and another factor of two from the construction of $\partial I^2/\partial r$. I can’t find the discrepancy.

**Solovév fields:** Here’s my form for the Solovév magnetic fields (poloidal part only):

$$B(r, \phi, z) = \psi_0 \left( \frac{8r\alpha^2z}{r_0^4}, 0, \frac{4r^2}{r_0^4} - \frac{8\alpha^2z^2}{r_0^4} \right)$$

From here you just have to remember that $B_x = B_r \cos \phi = B_r (x/r)$, etc. You also need to add in the $\phi$ field (due to the wire). Then use your favorite $x, y, z$ Runge-Kutta solver.

**MHD turbulence:** We saw that given the caveat that $E(k) \propto \epsilon^\alpha k^\beta$ we find that $E(k) = C \epsilon^{2/3} k^{-5/3}$. Abram asked about the effect of the largest scale $L_0$. The idea of Kolmogorov turbulence is that it applies to the “inertial” range... this is where the scale is small enough that the effects of the walls can be ignored (no $L_0$ dependence) but not so small that dissipation effects are important (no $\nu$ dependence). It’s a pretty particular set of conditions but if the scale of the system is large (like the solar wind or the ocean) then there are enough decades of $k$-space for an inertial range to exist. I put a link to that Berkeley website on our website.

Some solutions...

**Goldston 8.1** This makes more sense if you notice that $1 + \rho/\epsilon_0 B^2 \gg 1$ is the same as $v_{Alf}/c \ll 1$ (ie the phenomenon is slow).

**Goldston 8.3** I get $R_m = 10^{12–15}$ for corona and $R_m = 10^{8–10}$ for tokamak. Note that the corona is “only” about 100 eV so it’s conductivity is more like stainless steel. A tokamak at 10 keV is more like copper.

**Goldston 8.2** The idea is to dot the equation of motion with $u$:

$$u \cdot \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = \sigma E + J \times B - \nabla P$$

and eventually end up with the energy equation:

$$\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \frac{P}{\gamma - 1} + \frac{\epsilon_0 E^2}{2} + \frac{B^2}{\mu_0} \right) + \nabla \cdot \left( \frac{\rho u^2}{2} + \frac{E \times B}{\mu_0} + u - \gamma P \right) = 0$$

which when integrated over a large volume reduces to Goldston’s result (ie the surface terms vanish). Goldston does the kinetic energy and pressure terms. The best approach for the others is to use the ideal MHD Ohm’s law and to keep the displacement current (Chris’ idea). Looking at $u \cdot (J \times B)$ first...

$$u \cdot (J \times B) = (B \times u) \cdot J = (E - \eta J) \cdot J$$
The last term gives dissipation $\eta J^2$, let’s hold it in reserve for now.

\[
E \cdot J = E \cdot \frac{1}{\mu_0} \left( \nabla \times B - \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \right)
\]

Now notice the following:

\[
\frac{1}{\mu_0} \nabla \cdot (E \times B) = \frac{1}{\mu_0} (B \cdot \nabla \times E - E \cdot \nabla \times B)
\]

(this is just the divergence of the Poynting flux). From this we find:

\[
E \cdot J = \frac{1}{\mu_0} \left( B \cdot \nabla \times E - \nabla \cdot (E \times B) - \mu_0 \epsilon_0 E \cdot \frac{\partial E}{\partial t} \right)
\]

Now we invoke Faraday’s law and notice that $B \cdot dB/dt = 0$. Therefore:

\[
\frac{\partial}{\partial t} \left( \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) + \eta J^2 + \nabla \cdot \left( \frac{E \times B}{\mu_0} \right) = 0.
\]

Now if we let $\eta \to 0$ and do a volume integral over a closed box, we get Goldston’s result. As for the $\sigma E$ term:

\[
u \cdot (\sigma E) = \sigma \nu \cdot (-\nu \times B + \eta J) = \sigma \eta \nu \cdot J.
\]

If $\eta \to 0$, then this term doesn’t contribute but I don’t know how to justify it with $\eta J^2$ from above (if $\eta$ is finite). This should work out however.