General: The next step is to morph the two fluid equations from Goldston ch 6-7 into a set of equations governing a single electrically conducting fluid. This is called magnetohydrodynamics or MHD (sometimes hydromagnetics). It’s important to recognize exactly what physics is neglected in the MHD equations (electron inertia, displacement current, pressure tensor, etc). So phenomena that depend on particulars of the distribution function (say with multiple temperatures $T_{e,i}, T_{||,\perp}$) or involve large Larmor orbits or occur at very high frequency may not fit the MHD model. Nonetheless, the MHD model explains a lot of low frequency, large scale phenomena very well. In fact, MHD works in some instances that one might not think it should, eg collisionless plasmas (in which $\lambda_{MFP} \approx L$) or weakly ionized plasmas (in which $\nu_{e,N}$ is appreciable, see Bellan problem 2.8).

Required Reading: Goldston, chapters 8 and 9.

Supplementary Reading: Check out Bellan chapter 8 (I made extra copies of this important chapter), and Krall/Trivelpiece chapter 3.

Important Concepts: MHD equations, “frozen-in flux” (sec 8.4,5), magnetic Reynolds number, static equilibria (Bennett pinch), the concept of $\beta$, the magnetic induction equation (8.44), the three dimensional Grad-Shafranov equation (see Bellan).

Presentations: There are enough here for two weeks (6 topics to choose from). Let’s divide them up now so folks can have a head start. I’d say everyone should do what they can for each one... the point person to lead each discussion is indicated.

Derivation of MHD (Andrew): Goldston does a good job but have a look at the development in Krall/Trivelpiece (chapter 3). Krall is very careful to lay out the assumptions of MHD and to point out the ramifications of each assumption. Check out his discussion of “MHD ordering” as opposed to finite Larmor radius or “FLR” ordering (there’s a 1962 PRL about this).

Bennett (or Z) pinch (Mike L): Goldston does a fine job here too (he calls it a “cylindrical pinch”) but see the development in Jackson Classical Electrodynamics (section 10.5) also in Bellan chapter 8 for a different take. Compare Goldston’s pinch condition 9.14 to Bellan’s 8.29 and Jackson’s 10.45. This is the simplest one dimensional equilibrium and is a good intuition builder as to just what it might take to make a fusion reactor
work. The original Bennett article is from Physical Review 1934! Maybe try to track it down.

**Frozen-in flux (reading):** Goldston’s presentation is pretty clear. If everyone’s comfortable with the derivation then we could skip a presentation. Someone could look for other ways to show “flux freezing”... most of the derivations I know about are just like Goldston’s.

**Grad-Shafranov equation and Solovéy equilibrium (Abram):**

Look at Bellan chapter 8. A key point is that very generally, magnetic fields can be written in the following way:

\[
B = \frac{1}{2\pi} (\nabla \psi \times \nabla \phi + \mu_0 I \nabla \phi)
\]

where \(\nabla \phi = \hat{\phi}/r\). The first term describes the poloidal magnetic field \((\psi(r, z)\) is the poloidal flux) and the second term describes the toroidal magnetic field (swirling in the \(\hat{\phi}\) direction, \(B_{tor} = \mu_0 I/2\pi r\)). If you work out pressure balance \((J \times B = \nabla P)\) and assume cylindrical axisymmetry you get the Grad-Shafranov equation:

\[
\nabla \cdot \left( \frac{1}{r^2} \nabla \psi \right) + 4\pi^2 \mu_0 P' + \frac{\mu_0^2}{r^2} I' = 0
\]

The interesting thing is that we started with a 3D vector system involving \(B, v,\) and \(P\) (7 variables) and reduced it to a one dimensional scalar equation involving \(\psi\). Also interesting is that \(\psi\) is both an independent and a dependent variable, ie we’re taking spatial gradient of \(\psi\) but \(P'\) and \(I'\) refer to derivatives taken with respect to the variable \(\psi\). Think of \(\psi\) as a parametric label. A simple analytic solution is called the Solovéy equilibrium.

**For seminar 5 (get a head start!):**

**Particle orbits in realistic equilibria (Solovéy) (Abram):**

The analytical Solovéy equilibrium has the form:

\[
\psi(r, z) = \psi_0 \frac{r^2}{r_0^2} (2r_0^2 - r^2 - 4\alpha^2 z^2).
\]

Here \(\alpha\) is like the ellipticity and \(r_0\) is the magnetic axis (where \(\psi\) is a maximum). The magnetic field comes from

\[
B = \frac{1}{2\pi} (\nabla \psi \times \nabla \phi + \mu_0 I \nabla \phi)
\]

where \(I\) is just a constant (like a wire along the \(z\) axis). Plot surfaces of constant \(\psi\) then compute \(B\) and integrate some particle orbits. Do the orbits tend to stay near flux surfaces? This is Bellan’s problem 8.1. Also check

**MHD turbulence (Matt/Chris):** This is a huge subject that has implications in astrophysical plasmas as well as magnetic confinement fusion. Magnetic turbulence can lead to a random walk of a particle’s guiding center and therefore enhanced particle transport in a fusion device. Turbulence can also lead to “self-organization” and the assembly of large scale magnetic structures. There are a lot of tacks researchers have taken to study turbulence. One (that I know a little about) has to do with a statistical treatment of the MHD equations in Fourier space. The idea is that if we consider a finite system perhaps driven at some large scale \( (k_0 = 2\pi/L) \), then the disturbance can propagate to smaller scales (larger \( k \)) via a non-linear “cascade”. Energy is ultimately dissipated at small scales where \( \nabla^2 B \) is large (ie large \( k^2 \))... see the magnetic induction equation 8.44). There are also instances in which turbulence can support an “inverse-cascade” to larger scales (smaller \( k \)). This “inverse cascade” is related to the dynamo problem. I have some excellent typed lecture notes from David Montgomery as well as other references to get you started.

**Assigned Problems:** Goldston 8.1, 8.3 (not many good, concise HW problems out there on this topic)... maybe try your hand at more than one of the above presentations.

**Additional Problems:** Bellan’s problem 2.8 looks interesting (about how MHD works in weakly ionized plasmas).

**Break:** mb