Plasma gun notes

Here are some notes based on an idea of Paul Bellan's (see his Spheromak book for more details). It's a nice demonstration of Hamilton's equations of motion. It also shows how magnetic flux can be a canonical momentum variable... the electrodynamics comes for free. The equations of motion are not analytically soluble (I think) but shouldn't be hard to solve numerically. Give it a try (with Mathematica or IDL)!

1. Hamilton's equations:

The Hamiltonian is formally:

$$H = \sum_j p_j \dot{q}_j - L = T + U$$

where L is the Lagrangian, p_j are the canonical momenta, and q_j are the canonical coordinates. The coordinates and momenta are connected through the Lagrangian:

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Hamilton's equations are

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \quad \frac{\partial H}{\partial p_j} = \dot{q}_j$$

The Hamilton approach is elegant and exposes conservation laws.

2. Plasma gun:

We can model the plasma gun as coaxial inductor with inductance $L(x) = L_0 + \mathcal{L}x$ and a fixed capacitor on the back end C. $\mathcal{L}$ is an inductance per unit length and L_0 is the inductance of the system before the spheromak starts to move. The spheromak is a sliding short of mass m impaled on the center electrode. The potential energy of the system (before current starts to flow and mass begins to move) is:

$$U = \frac{q^2}{2C}$$

The "kinetic" energy of the system is:

$$T = \frac{p^2}{2m} + \frac{1}{2}L(x)\dot{q}^2$$

The Lagrangian of the system is then:

$$L = T - U = \frac{p^2}{2m} + \frac{1}{2}L(x)\dot{q}^2 - \frac{q^2}{2C}$$

We can get the appropriate canonical momenta from:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \;\text{ and }\; p_q = \frac{\partial L}{\partial \dot{q}} = L(x)\dot{q}$$

Note that the general momentum p_q isn't a momentum at all! It's the magnetic flux behind the spheromak ($\Psi \equiv L(x)\dot{q}$).

You can write the Hamiltonian for the system in terms of the canonical variables:

$$H(x, p_x, q, p_q) = \sum_j p_j \dot{q}_j - L = \frac{p_x^2}{2m} + \frac{p_q^2}{2L(x)} + \frac{q^2}{2C} = \frac{p^2}{2m} + \frac{1}{2}L(x)\dot{q}^2 + \frac{q^2}{2C}$$

There are 4 Hamilton's equations using the 2 canonical coordinates (x, q) and the associated canonical momenta $(p, \Psi \equiv L(x)\dot{q})$:

$$\frac{\partial H}{\partial x} = -\frac{p_q^2 \mathcal{L}}{2L^2(x)} = -\frac{1}{2}\mathcal{L}\dot{q}^2 = -\dot{p} = -m\ddot{x} \;\; \text{(main eq of motion)}$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x} \;\; \text{(def of linear mom)}$$

$$\frac{\partial H}{\partial q} = \frac{q}{C} = -\dot{p}_q = -\dot{\Psi} \;\; \text{(Faraday, def of mom canonical to q)}$$

$$\frac{\partial H}{\partial p_q} = \frac{p_q}{L(x)} = \frac{L(x)\dot{q}}{L(x)} = \dot{q} \;\; \text{(tautology)}$$

The initial conditions are the spheromak at $x = 0$ and the capacitor fully charged. As the capacitor drains, current flows and so a force is applied to the spheromak.