More on the mirror force ($F = \mu \nabla B$)

We have seen that the magnet moment of a proton in a magnetic field is: $\mu = \frac{W_\perp}{B}$, and we saw that if there's a region of lower field between two magnetic mirrors, then there's a criterion for particles being in a loss cone (or escaping):

$$\frac{v_\parallel}{v_\perp} > \left(\frac{B_{max}}{B_{min}} - 1\right)^{1/2} \equiv (R_m - 1)^{1/2}$$

This loss cone formula comes from noting that

$$\frac{W_\parallel}{W} = \frac{v_\parallel^2}{v_\perp^2 + v_\parallel^2}$$

Two things we talked about in the tutorial session were the origin of the mirror force, and from that, we can prove in a differnet way that μ is conserved.

Mirror force (Nicholson): As the orbiting proton drifts into a region of higher magnetic field, it encounters an axial force due to the small radial component of the magnetic field. When the field lines are straight, the only magnetic force present is the one that makes the orbit circular. The axial component of the force is: $F = q(v \times B) = qv_\perp B_r$. We can estimate B_r from the little-used Maxwell equation $\nabla \cdot B = 0$. In cylindrical coordinates, we have:

$$\nabla \cdot B = \frac{\partial B_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rB_r) = 0$$

So, assuming B_z doesn't change much:

$$rB_r = -\int r\frac{\partial B_z}{\partial z} dr \cong -\frac{r^2}{2}\frac{\partial B_z}{\partial z}$$

We get for the axial force (since $r = r_i$):

$$F_z = -qv_\perp \frac{r}{2}\frac{\partial B_z}{\partial z} = -\frac{qv_\perp}{2}\frac{Mv_\perp}{qB}\frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z}$$

μ conservation (Chen): From that force law and conservation of energy, we can show the other way μ is conserved. Newton II for the force law is:

$$M\frac{dv_\parallel}{dt} = -\mu\frac{\partial B_z}{\partial z}$$

If we multiply both sides by v_z and integrate we get:

$$Mv_{\parallel}\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B_z}{\partial z}\frac{\partial z}{\partial t} = -\mu\frac{dB_z}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}Mv_{\parallel}^2\right) = -\mu\frac{dB_z}{dt}$$

From energy conservation, we have:

$$\frac{d}{dt}\left(\frac{1}{2}Mv_{\parallel}^2 + \frac{1}{2}Mv_{\perp}^2\right) = \frac{d}{dt}\left(\frac{1}{2}Mv_{\parallel}^2 + \mu B\right) = 0$$

Since $d/dt(\mu B) = \mu dB/dt + B d\mu/dt$ and substituting from the equation above, we have:

$$-\mu\frac{dB_z}{dt} + \frac{d}{dt}\left(\mu B\right) = 0 \rightarrow \frac{d\mu}{dt} = 0$$