Adiabatic invariants (μ and J)

The deep theory behind adiabatic invariants and why they are important for equations of state comes from Hamiltonian theory in advanced mechanics. This is covered in more advanced plasma texts like Bellan or Fitzpatrick. The Poincare invariant looks like: $\mathcal{I} = \oint p\ dq$, where p and q are generalized coordinates in Hamilton's equations.

First adiabatic invariant μ: The magnetic moment of a loop of wire is just $\mu = IA$. For a proton orbiting a magnetic field the current is $I = q/t = q\omega_{ci}/2\pi$. A few things we need are:

$$A = \pi r_i^2, \omega_{ci} = qB/M, r_i = \frac{Mv}{qB} = \frac{v}{\omega_{ci}}$$

So we have:

$$\mu = IA = \frac{\pi r_i^2 q\omega_{ci}}{2\pi} = \frac{1}{2}\frac{v^2 q\omega_{ci}}{\omega_{ci}^2} = \frac{1}{2}\frac{v^2 q}{\omega_{ci}} = \frac{1}{2}\frac{Mv_\perp^2}{B}$$

There are several ways to show that μ is conserved. First, since charged particle orbits conserve magnetic flux, we have that:

$$\Phi = BA = B\pi r_i^2 = \frac{B\pi v^2}{\omega_{ci}^2} = \frac{B\pi v^2 M^2}{q^2 B^2} \sim \frac{Mv^2}{B}$$

so the magnetic moment is also conserved.

The other approach is that if you believe that $\mathcal{I} = \oint p\ dq$ is conserved, then for us (using p as the angular momentum and q as the angle)

$$\mathcal{I} = \oint Mvr\ d\theta = \oint \frac{Mv^2}{\omega_{ci}}\ d\theta \sim \frac{Mv^2}{B}$$

Finally, we can show directly that μ is conserved (see Goldston and Rutherford):

$$\frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = \frac{1}{B}\frac{dW_\perp}{dt} - \frac{W_\perp}{B^2}\frac{dB}{dt} = 0$$

Second adiabatic invariant J:

Using $\mathcal{I} = \oint p\ dq$, but this time $p = Mv_\parallel$ is the axial momentum and q is the axial distance

$$\mathcal{I} = \oint Mv_\parallel dz \sim v_\parallel L$$

Magnetic mirrors: The constancy of μ leads to an important process called magnetic mirroring. The idea is that protons with a large pitch angle (ie have mostly perpendicular velocity with respect to the magnetic field) will be trapped between regions of higher magnetic field. This is the physics behind the magnetic mirror confinement device, Fermi acceleration in astrophysical plasmas, and the Van Allen radiation belts in the magnetosphere.

The total (conserved) energy of the proton is:

$$W = W_{\|} + W_{\perp} = \frac{1}{2}Mv_{\|}^2 + \mu B$$

As the proton streams towards the mirror, since μ is conserved $W_{\perp}$ will spin up in proportion to B and $W_{\|}$ will drop. Protons with $\mu = W/B_{max}$ will be just marginally trapped. For those particles at the midplane ($B = B_{min}$ there):

$$W_{\perp}(midplane) = \mu B_{min} = \frac{WB_{min}}{B_{max}}$$

$$\frac{W_{\|}(midplane)}{W} = \frac{W - W_{\perp}(midplane)}{W} = 1 - \frac{B_{min}}{B_{max}}$$

This defines a "loss-cone" in velocity space:

$$\frac{v_{\|}}{v_{\perp}} > \left(\frac{B_{max}}{B_{min}} - 1\right)^{1/2} \equiv (R_m - 1)^{1/2}$$

Equations of state: The simple MHD adiabatic equation of state is appropriate for a collisional plasma (or gas) where the mean free path is short compared to a Larmor orbit (or $\omega_{ci}\tau_{coll} < 1$). In this case the magnetic field plays no role:

$$\frac{d}{dt}\left(\frac{P}{n^{\gamma}}\right) = 0$$

The CGL or double adiabatic theory has two formulae. These are appropriate if the plasma is magnetized (ie the magnetic field plays a role), and the protons execute several orbits before colliding ($\omega_{ci}\tau_{coll} > 1$):

$$\frac{d}{dt}\left(\frac{P_{\perp}}{nB}\right) = 0$$

$$\frac{d}{dt}\left(\frac{P_{\|}B^2}{n^3}\right) = 0$$