Physics 138
Plasma Physics 2003
Seminar 1 presentation (mb)

ω_{pe}, ν_c, and Λ (mb): Show the connection among Debye length λ_D, plasma parameter Λ, plasma frequency ω_{pe}, and collision frequency ν_c.

Saha equation: Before we get started, it's useful to consider the Saha equation which tells us the ionization fraction for a gas in thermal equilibrium at temperature T:

$$\frac{n_e}{n_{neutral}} \cong 2.4 \times 10^{21} \frac{T_K^{3/2}}{n_e} e^{-U_i/kT}$$

where T is in Kelvin, the densities are in particles per m^3, and U_i is the ionization potential. For air in this room, T = 300K, $n_{neutral} = 3 \times 10^{25}\ m^{-3}$, and $U_i = 14.5\ eV$ (nitrogen). We find that $n_e/n_{neutral} = 10^{-122}$. Since there are only a few 1000 Avogadro numbers worth of particles in the room (maybe 10^{27}), there are effectively **no** electrons from thermal ionization. The derivation is here: http://scienceworld.wolfram.com/physics/SahaEquation.html

Debye length λ_D: First let's write down the Poisson equation for the electrostatic potential (MKS):

$$\nabla^2 \phi = -\rho/\epsilon_0 = e(n_e - n_i)/\epsilon_0.$$

Now assume that the ions and electrons are separately in thermal equilibrium... ie they each have a temperature and a Boltzmann factor like $n = n_0\ exp(e\phi/kT)$. If $e\phi/kT \ll 1$ then we can expand the exponential and find:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = \frac{n_0 e^2}{k\epsilon_0}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)\phi.$$

We define the Debye length for each species

$$\lambda_D = \left(\frac{\epsilon_0 kT}{n_0 e^2}\right)^{1/2} (MKS) = \left(\frac{kT}{4\pi n_0 e^2}\right)^{1/2} (CGS)$$

and the total Debye length as the sum of the inverse squares.

From here on out, we'll incorporate Boltzmann's constant into the temperature T and talk about temperature like an average particle energy measured in electron volts. Also, we'll combine constants like k, ϵ_0, e and write $\lambda_D = 740(T/n)^{1/2}\ cm$ where T is measured in eV and n is in units of particles per cm^3. So a plasma at 1 eV and 1 cm^{-3} (say the solar wind) has a Debye length of 740 cm.

Now we can solve for the potential:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{\phi}{\lambda_D^2}.$$

The solution is

$$\phi(r) = \frac{q}{r}e^{-r/\lambda_D}.$$

The interpretation is that if the Debye length is large (say very low plasma density like the air in this room) then the exponential is about unity and the potential falls off like $1/r$. If the Debye length is short (low temperature and high density like in a metal), then point charges are immediately shielded and the potential drops to zero in a few Debye lengths ($\lambda_D \ll 1\mathring{A}$ in a metal). When we say that free charges reside on the surface of a metal, we should really say that free charges reside a few Debye lengths from the surface of a metal.

plasma parameter Λ: A useful dimensionless number is the number of particles in a Debye cube (or sphere). This is called the plasma parameter:

$$\Lambda = n\lambda_D^3 \gg 1$$

and the requirement that there are lots of particles in a Debye cube is one of the definitions of a plasma (for statistical reasons if nothing else). It turns out that Λ has several other interpretations (see below and other presentations). Numerically we get $\Lambda = 4 \times 10^8\ T^{3/2}n^{-1/2}$.

plasma frequency ω_{pe}: Consider a slab 'o plasma (ions and electrons in equal numbers) with area A and thickness L (ie volume $= AL$). If we displace the electrons in the slab a small distance δ from their equilibrium positions (say to the right), then we'll expose a layer of ions of thickness δ on the left. An electric field of magnitude $E = \sigma/\epsilon_0$ now points to the right where the surface charge $\sigma = ne\delta$. The force acting on the electron fluid is

$$F = Q_{tot}E = (neAL)\left(\frac{ne\delta}{\epsilon_0}\right) = \frac{n^2e^2AL\delta}{\epsilon_0} = M_{tot}\ddot{\delta} = nmLA\ddot{\delta}.$$

From which we can write:

$$\ddot{\delta} = \left(\frac{ne^2}{m\epsilon_0}\right)\delta.$$

We can immediately identify the plasma frequency

$$\omega_p = \left(\frac{ne^2}{m\epsilon_0}\right)^{1/2}(MKS) = \left(\frac{4\pi ne^2}{m}\right)^{1/2}(CGS).$$

Numerically, $\nu_{pe} = 9000n_e^{1/2}\ Hz$ so a typical laboratory plasma at $10^{10}\ cm^{-3}$ wiggles at 1 GHz. The ionosphere has an electron density of $10^5\ cm^{-3}$ so its plasma frequency is about 3 MHz which is intermediate between AM and FM radio.

collision frequency ν_c: Imagine a charged particle (mass m, charge q, velocity v_0) approaching another charged particle at rest (mass $M \gg m$, charge q_0). If v_0 is small, then the incoming particle won't be able to get too close to the target particle without getting deflected say 90^o. The more kinetic energy the incoming particle has, the closer it can approach the target particle so we see there's a scale we can associate with the energy:

$$\frac{mv_0^2}{2} = \frac{qq_0}{4\pi\epsilon_0\delta}.$$

The parameter $\delta = \frac{2qq_0}{4\pi\epsilon_0 mv_0^2}$ is sometimes called the Landau length.

If we have a population of particles of density n_0, velocity v_0, adn charge e all heading for our target particle (also charge e), the rate at which they get scattered (say 90^o or more) is roughly the flux of particles that pass within a radius δ of the target:

$$\nu_c = (\pi\delta^2)nv_0 = \pi\frac{4e^4}{(4\pi\epsilon_0)^2m^2v_0^4}nv_0 = \frac{e^4n}{m^2v_0^3(4\pi\epsilon_0^2)}.$$

If you do this more carefully (considering small angle collisions) you get an extra factor of 2 $ln(\Lambda) \cong 20$, also in CGS the factor of $(4\pi\epsilon_0)$ downstairs is replaced by a factor of (4π) upstairs. The important feature to notice is that the collision frequency goes like v_0^{-3} or, if the particles are a thermal distribution, like $T^{-3/2}$.

The connection: Finally, the point of all this is to show the relation among these parameters. Look at the ratio of the plasma frequency to the collision frequency:

$$\frac{\omega_p}{\nu_c} = \left(\frac{ne^2}{m\epsilon_0}\right)^{1/2}\left(\frac{m^2v_0^3(4\pi\epsilon_0^2)}{e^4n}\right) = \frac{4\pi\epsilon_0^{3/2}m^{3/2}v^3}{n^{1/2}e^3}.$$

Notice now that the plasma parameter can be written:

$$\Lambda = n\lambda_D^3 = n\left(\frac{\epsilon_0 T}{ne^2}\right)^{3/2} = \frac{\epsilon_0^{3/2}m^{3/2}v^3}{n^{1/2}e^3}$$

where we note that

$$\frac{m < v^2 >}{2} = \frac{3T}{2} = \frac{3mv^2}{2}.$$

We find that

$$\frac{\omega_p}{\nu_c} = 4\pi\Lambda.$$

The interpretation here is that a plasma will oscillate many times at ω_p before it is damped by collisions. This really emphasizes the point that plasmas are dominated by collective effects (ω_p) rather than single particle effects (ν_c).