WHAM mirror physics notes  
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Preliminaries: WHAM is an axisymmetric, super-conducting magnetic mirror machine (Baldwin 1977, Post 1987) being built at the University of Wisconsin (Figure 1). The super-conducting mirror magnets (manufactured by Commonwealth Fusion Systems, CFS) will operate at 17 T (2 kADC, 3 MJ each), while the two central cell magnets (copper, from W7-A in Germany) will operate at 0.8 T. The field at the end wall is 0.05 T.

The mirror ratio for phase-II will be $R_m = 17/0.8 \approx 20$. ECH will heat electrons at the 4 T layer (110 GHz) at 1 MW. NBI will provide sloshing ions at 25 keV and 2 MW. Fast wave ion acceleration will be at the 2nd harmonic of D at 2 T (26 MHz) at 1 MW. The central cell plasma will have $R = 0.08 \, m, n_e = 3 \times 10^{19} \, m^{-3}, T_e = 1.3 \, keV, \langle E_i \rangle = 20 \, keV, \beta = 0.4$. Plasma length is about $L = 1$ meter, mirror-to-mirror length is 2 meters (350 ton attractive force), and overall machine length is about 5 meters. Plasma volume is about 20 liters (same as SSX, but $N = 6 \times 10^{17}$). The throat has $r = 0.0275 \, m$ (about an inch). Pulse length will be about 20 ms.

Figure 1: Rendering of WHAM device. Central cell is green. Superconducting mirror coils in brown. Copper central cell coils in yellow. ECH beam in purple with steering mirrors (right). ICF antenna strap in brown (left).
Figure 2: Model field lines for WHAM. Model described below. Dimensions are in meters. Note the “bad” curvature in the central cell.

**μ conservation:** The magnetic moment of a loop of wire is just \( \mu = IA \).

For a proton orbiting a magnetic field the current is \( I = q/t = q\omega_{ci}/2\pi \). A few things we need are:

\[
A = \pi r_i^2, \quad \omega_{ci} = qB/M, \quad r_i = \frac{Mv}{qB} = \frac{v}{\omega_{ci}}
\]

So we have:

\[
\mu = IA = \frac{\pi r_i^2 q\omega_{ci}}{2\pi} = \frac{1}{2} \frac{v^2 q\omega_{ci}}{\omega_{ci}^2} = \frac{1}{2} \frac{v^2 q}{\omega_{ci}} = \frac{1}{2} \frac{Mv^2}{B}
\]

There are several ways to show that \( \mu \) is conserved (see Goldston and Rutherford, which begins):

\[
\frac{d\mu}{dt} = \frac{d}{dt} \left( W_\perp \frac{B}{B} \right) = \frac{1}{B} \frac{dW_\perp}{dt} - \frac{W_\perp dB}{B dt} = 0
\]

From Chen’s derivation, the force law is:

\[
M \frac{dv_{||}}{dt} = -\mu \frac{dB_z}{dz}
\]

If we multiply both sides by \( v_z \) and integrate we get:

\[
Mv_\parallel \frac{dv_{||}}{dt} = -\mu \frac{\partial B_z}{\partial z} \frac{dz}{dt} = -\mu \frac{dB_z}{dt}
\]

\[
\frac{d}{dt} \left( \frac{1}{2} M v_{||}^2 \right) = -\mu \frac{dB_z}{dt}
\]

From energy conservation, we have:

\[
\frac{d}{dt} \left( \frac{1}{2} M v_{||}^2 + \frac{1}{2} M v_{\perp}^2 \right) = \frac{d}{dt} \left( \frac{1}{2} M v_{||}^2 + \mu B \right) = 0
\]
Since \( d/dt(\mu B) = \mu dB/dt + Bd\mu/dt \) and substituting from the equation above, we have:

\[
-\mu dB_z/dt + d/dt(\mu B) = 0 \rightarrow d\mu/dt = 0
\]

**Mirror physics:** From above the magnetic moment of a proton in a magnetic field is: \( \mu = \frac{W_\perp}{B} \), and if there’s a region of lower field between two magnetic mirrors, then there’s an important criterion for particles being in a loss cone (or escaping):

\[
\frac{v_\parallel}{v_\perp} > \left( \frac{B_{\text{max}}}{B_{\text{min}}} - 1 \right)^{1/2} \equiv (R_m - 1)^{1/2}
\]

This loss cone formula comes from noting that:

\[
\frac{W_\parallel}{W} = \frac{v_\parallel^2}{v_\perp^2 + v_\parallel^2}
\]

And using energy conservation:

\[
W = W_\parallel + W_\perp = \frac{1}{2} M v_\parallel^2 + \mu B
\]

so all particles with \( \mu \geq W/B_{\text{max}} \) are trapped. If \( \mu = W/B_{\text{max}} \), the particle is marginally trapped. So for marginally trapped particles at the midplane we have:

\[
W_\perp = \mu B_{\text{min}} = W \frac{B_{\text{min}}}{B_{\text{max}}} \rightarrow \frac{v_\perp^2}{v^2} = \frac{B_{\text{min}}}{B_{\text{max}}}
\]

\[
W_\parallel = W \left( 1 - \frac{B_{\text{min}}}{B_{\text{max}}} \right) \rightarrow \frac{v_\parallel^2}{v^2} = 1 - \frac{B_{\text{min}}}{B_{\text{max}}}
\]

So the loss cone formula becomes:

\[
\frac{v_\parallel}{v} > \left( 1 - \frac{B_{\text{min}}}{B_{\text{max}}} \right)^{1/2}
\]

or equivalently,

\[
\frac{v_\parallel}{v_\perp} > \left( \frac{B_{\text{max}}}{B_{\text{min}}} - 1 \right)^{1/2}
\]

In velocity space, the angle between the paraxial direction and the pitch of the marginally trapped protons is given by:
\[ \sin^2 \theta = \frac{1}{R_m} \approx \theta^2 \]

This is because \( \mu \) is conserved and there is no \( v_\parallel \) at the turning point, \( B_{\text{max}} \) so:

\[
\mu = \frac{mv_{\text{min,} \perp}^2}{2B_{\text{min}}} = \frac{mv_{\text{max,} \perp}^2}{2B_{\text{max}}} \rightarrow \frac{B_{\text{min}}}{B_{\text{max}}} = \frac{v_{\text{min,} \perp}^2}{v_{\text{max,} \perp}^2} \approx \frac{v_{\text{min,} \perp}^2}{v_{\text{max}}^2}
\]

But \( \sin \theta \equiv v_{\perp}/v \), so \( \sin^2 \theta = 1/R_m \). At \( R_m \gg 1 \), \( \theta^2 \approx 1/R_m \).

The axial force is given by:

\[
F_z = -\mu \frac{\partial B_z}{\partial z} = -\mu \cdot \nabla B
\]

The axial ion loss rate (in the GDT at least) is the thermal ion flux mediated by the mirror ratio:

\[
\nu_{\text{GDC}} = \frac{nv_i \pi R^2}{R_m}
\]

So the plasma lifetime can be estimated by dividing the total particle number by this rate:

\[
\tau_{\text{GDC}} \approx \frac{R_m n \pi R^2 L}{nv_i \pi R^2} = \frac{R_m L}{v_i}
\]

This means you want large mirror ratios, and large separation between the mirrors.

Finally, and interestingly, if the ratio of the field at the end wall (0.05 T) to the field at the mirror throat (17 T) is less than the square root of the mass ratio:

\[
\frac{0.05}{17} \approx \frac{1}{340} \leq \sqrt{\frac{m_e}{M_p}} = \frac{1}{43}
\]

then cold end-cell electrons can’t get into the central cell (Ryutov 2000). This means you want large end cells with the end plate far from the mirrors (see Figure 3).
Three issues: There are three big issues with mirror machines related to the particles (electrons and protons), and the MHD fluid character of the plasma (Simonen 2016). For electrons, the issue is parallel heat conduction along nearly straight magnetic field lines. For ions, it’s the kinetic effects of a non-Maxwellian velocity distribution. For the MHD fluid, it’s ballooning and interchange instabilities at the edge with bad curvature.

1. Electrons (parallel heat conduction): At high mirror ratio, the small population of particles in the (two) upstream and downstream loss cones (ions or electrons) is given by the integral over the unit sphere:

\[ 2 \frac{4\pi}{3} \int_0^\theta \sin \theta \, d\theta \approx 2 \frac{4\pi}{3} \frac{\theta^2}{2} = N_{tot} \theta^2 \approx \frac{N_{tot}}{R_m} \]

This means that only a few percent of the particles are in the loss-cone for \( R_M \gg 1 \). Nonetheless, if the central cell density is \( n_e = 3 \times 10^{19} \text{ m}^{-3} \), there can be a density of \( n_{pass} = 10^{17} \text{ m}^{-3} \) collision-less, hot, parallel electrons tied to each end cell. This population can act as a heat sink directly connecting the hot central cell to the far wall (at room temperature), where a 1 keV electron is exchanged for a 1 eV secondary electron.

Fortunately, electron confinement is strongly modified by the ambipolar potential, \( \phi_M \) (Coensgen 1980). The physics is that since the electrons are less massive than the protons \( (m_e/M_p = 1/1836) \), they move \( \sqrt{M/m} \approx 43 \) times faster than protons for the same temperature \( (Mv_i^2 = mv_e^2) \). A small population of electrons leaves the device, leaving behind a positive potential \( e\phi_M \approx kT_e \ln(n/n_0) \) that holds the electrons in. The ambipolar electric field points axially outward through the mirror throats. In the end, because the plasma remains quasineutral, and because there can be no net current from
the device, the ion and electron fluxes balance: $n_e c_S$, where $c_S = \sqrt{kT_e/M}$ the sound speed.

If the potential holding in the electrons (and accelerating the ions out of the mirror) has the form: $e\phi_{ion} = e\phi_{plug} - e\phi_{center} = kT_e \ln(n_p/n_c)$, then the ion lifetime can be written: $n_c \tau_i = n_c \tau_{ii}(e\phi_i/kT_{ic}) \exp(e\phi_i/kT_{ic})$ (Pastukhov 1974, Yatsu 1979).

In the Gas Dynamic Trap (GDT) in Novosibirsk, the density is high enough so that the electron mean free path is less than the mirror-mirror distance ($\lambda_{MFP} \leq L$), (see Ivanov 2017 and Burdakov 2010). Except for a small population near the mirrors, the central cell plasma is nearly isotropic and Maxwellian in the GDC. The GDC has demonstrated $T_e \approx 1$ keV (Bagryan-sky 2015)

In WHAM, the electrons are lost at a rate identical to the ions due to ambipolarity. So, they are confined for many thermalization times and correspondingly should be close to isothermal. For subtle reasons related to pressure anisotropy affecting the equilibrium (Krasheninnikov 2000), the ambipolar potential in a mirror machine can be many times the electron temperature ($kT_e/e$). There is a prediction that we will have a $e\phi_M \sim 5 kT_e$ potential drop between the center and the end plates in the large expanders so that only electrons with energy $\geq 5 kT_e$ are able escape and this is a small number (Wetherton 2021).

Figure 4: Electron phase space of $v_{||}, v_\perp$ in WHAM. Electrons above the blue line are in the loss cone.
Figure 4 shows the electron phase space of \( v_{\parallel}, v_{\perp} \) in WHAM, with \( R_m = 20 \) and axial potential \( \phi_M = 1 \ T_e \). Axes are in units of electron thermal speed \( (\sqrt{kT/m}) \), and potential is in temperature units \( (\phi_M = 2e\phi_0/m) \). The simple model can be written:

\[
v_{\parallel}^2 = v_{\perp}^2(R_m - 1) + \phi_M
\]

2. Ions (loss cone distribution microinstabilities): Kinetic instabilities tap into the free energy associated with a loss-cone (ie non-equilibrium, non-Maxwellian) velocity distribution. One is the high-frequency convective instability (HFC). Those operate at very high frequency: \( \omega = \omega_{pe}k_{\parallel}/k \).

Another is the Alfvén ion cyclotron (AIC) which couples longitudinal Alfvén waves with transverse ion fluctuations and propagates along field lines. AIC operates at about \( \omega_{ci} \), rotates in the ion direction, is driven by large \( A = T_{\perp}/T_{\parallel} \) (actually \( \beta A^2 \)), and has low azimuthal mode number (Casper 1982).

The most dangerous is the drift-cyclotron loss-cone (DCLC) instability (Post and Rosenbluth 1966). DCLC also operates near \( \omega_{ci} \) with the electron drift frequency interacting with ions, and is driven by the loss-cone distribution, the density gradient, and finite \( \rho_i/R \leq 1 \). It is flute-like with \( k_{\parallel} = 0 \). The coupling condition is \( \omega_{ci} \approx \omega_{\text{drift}} \approx \omega_{pi}^2/\omega_{ci} \) or \( \omega_{ci} \approx \omega_{pi} \). Also, \( \omega/k_{\perp} = v_{Di} \) and \( k_{\perp}R = m \), with \( k_{\perp}\rho_i \geq 1 \). The growth rate is essentially the cyclotron frequency, so DCLC is almost always observed as a saturated state. It is similar to AIC in that it operates at about \( \omega_{ci} \), rotates in the ion direction, but is a different beast. DCLC killed the mirror program in the 1980’s.

The DCLC begins with the electron drift instability (in the electron diamagnetic direction) destabilized by a radial density gradient. If the perpendicular ion velocity distribution has a region of positive slope at velocities near the ion drift velocity, more ions give energy to the wave than gain energy from the wave due to the coupling of the wave to the ion-cyclotron motion. This condition, which occurs in a magnetic mirror with a loss-cone ion velocity distribution, modifies the drift-cyclotron mode to create the drift-cyclotron loss-cone mode (Ferron 1984, Kotelnikov 2017, Prikhodko 2020).

The fix for this is to put warm ions back in the loss cone, for example with sloshing ions generated by neutral beams directed at 45° with energy of 25 keV (see Figure 3, and Simonen, 1983). If \( \theta = \pi/4 \), then \( \sin^2\theta = 1/2 \) and the sloshing ions mirror at \( R_m = 2 \). Coupled to this is an idea to push beam ions to re-populate the loss cone using High-Harmonic Fast Magnetosonic Wave (HHFW, related to ICW, Ono 1995, Okada 2015). This will operate
at the 2nd harmonic of D at 2 T (26 MHz) at 1 MW. Typically, HHFW operates at $\omega/\omega_{ci} \approx 10$.

3. MHD fluid (ballooning and interchange): Because field lines in the central cell are necessarily concave inwards (see Figure 2), paraxial flows experience an effective gravity outwards and a Rayleigh-Taylor-like fluid instability (see Figure 2). This is the curvature-driven interchange instability (Kesner 2000), closely related to a ballooning instability, both leading to growing flute-type modes at the edge ($k_{||} \sim 0$) (see Connor 1978; Bilikmen 1997). These instabilities can be ameliorated by sheared transverse flows. Those are generated via vortex confinement in both GDC and WHAM (see Beklemishev 2010, Soldatkin 2017). GAMMA-10 has done something similar using off-axis ECH (Yoshikawa 2019).

This is related to, but different from, sheared flows in tokamaks to break up convective eddies for H-mode. The difference is that this is a strongly rotating, thin layer, just inside the limiter. The idea is that the rotating layer keeps the drift modes at marginal stability (ie small).

MHD stability has been a big issue since the inception of the mirror program. In the US, non-axisymmetric end mirrors (yin-yang and baseball) were used at TMX-U and MFTF-B at Livermore (1970s) to use pressure-weighted good field line curvature in the end cells. In Russia, axial current-carrying rods called Ioffe bars in a quadrupole pattern were used in the central cell. At the end of the US mirror program (mid-1980s), MHD stabilization was demonstrated in axisymmetric mirrors on Phaedrus (UW) using rf-ponderomotive pressure (Ferron 1983, Hershkowitz 1985), and Constance/Tara at MIT.

We will monitor coherent edge fluctuations in WHAM with azimuthal arrays of magnetic probes, and capacitively-coupled electrostatic probes (Tan 2017, and Wang 1987).

Field line model: The actual vacuum field for WHAM can be computed using the Biot-Savart law with knowledge of positions and currents of all field coils. A simple axisymmetric model can be constructed using the following flux function (see Figure 2):

$$\Psi(R, Z) = \frac{R^2 B_0}{2\pi \gamma} \left[ \frac{1}{1 + \left( \frac{Z-Z_m}{\gamma} \right)^2} + \frac{1}{1 + \left( \frac{Z+Z_m}{\gamma} \right)^2} \right]$$

where $R, Z$ are the radial and axial coordinates, $Z_m = 1 m$ is the mirror location, $B_0 = 6.5 T$ is the scaled field, $\gamma \sim 0.125$ is a curvature parameter. This model is surprisingly accurate. The flux function is like a potential, so the field components are calculated this way (Bellan, ch. 9):
\[
\mathbf{B}(R, Z) = \frac{1}{2\pi} (\nabla \Psi \times \nabla \phi) = -\frac{1}{R} \frac{\partial \Psi}{\partial Z} \hat{R} + \frac{1}{R} \frac{\partial \Psi}{\partial R} \hat{Z}
\]

I find:

\[
B_R(R, Z) = \frac{RB_0}{\pi \gamma^3} \left[ \frac{(Z - Z_m)}{\left[1 + \left(\frac{Z - Z_m}{\gamma}\right)^2\right]^2} + \frac{(Z + Z_m)}{\left[1 + \left(\frac{Z + Z_m}{\gamma}\right)^2\right]^2} \right]
\]

\[
B_Z(R, Z) = \frac{B_0}{\pi \gamma} \left[ \frac{1}{1 + \left(\frac{Z - Z_m}{\gamma}\right)^2} + \frac{1}{1 + \left(\frac{Z + Z_m}{\gamma}\right)^2} \right]
\]

Figure 5: Model B field vectors for WHAM using the model above. Arrows near the mirror throat correspond to 17 T. Dimensions are in meters.

**Neutronics:** With good D-D confinement in WHAM, we expect to see fusion reactions generating neutrons. The Lawson parameter will be \(nT\tau \approx 0.3 \times 10^{20} \text{ m}^{-3} \times 20 \text{ keV} \times 10^{-2} \text{ s} = 6 \times 10^{18} \text{ m}^{-3} \text{ keV} \text{s} \). There are two fusion reactions with D-D. One produces T, which will undergo D-T reactions:

\[
D + D \rightarrow T(1 \text{ MeV}) + p(3 \text{ MeV})(50\%)
\]

\[
D + D \rightarrow He^3(0.8 \text{ MeV}) + n(2.45 \text{ MeV})(50\%)
\]

\[
D + T \rightarrow He^4(3.5 \text{ MeV}) + n(14 \text{ MeV})
\]

The thermal reaction rate for D-D at about 20 keV is \(\langle \sigma v \rangle \approx 5 \times 10^{-18} \text{ cm}^3/\text{s} \) (NRL formulary). This is pretty small, but in WHAM the main reaction is from the sloshing ions that stagnate at \(R_m = 2\). The predicted reaction rate for DD is \(10^{13}/\text{s} \) and for DT is \(10^{15}/\text{s} \), so in a millisecond, we expect \(10^{10} \) DD neutrons for phase II (\(10^8 \) per millisecond in phase I).
DD neutrons can be useful in a $Q \leq 1$ fusion neutron source or FNS (say for radio-isotope production, Simonen 2013, Yurov 2016), or a $Q \gg 1$ power plant (Post 1987).

Figure 6: Rendering of WHAM device in Phase II. Louvered bias rings for vortex confinement are shown in the end cells. ICF antenna strap in brown (right). ECH launcher is on the left. Two copper magnets are shown at the midplane. Two superconducting magnets are shown at the mirror throats.
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