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Density Fluctuation Spectra in Magnetohydrodynamic Turbulence

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It is shown that within the framework of nearly incompressible magnetohydrodynamics, but not within that of neutral-fluid hydrodynamics, a $k^{-5/3}$ inertial-range wave number density fluctuation spectrum is to be expected at the same times that $k^{-5/3}$ kinetic energy and magnetic energy cascade spectra are present. A previous discrepancy between theory and observation in the local interstellar medium and solar wind is thereby resolved.

A recurrent but theoretically unexplained observation has been that of a $k^{-11/3}$ modal wave number spectrum, or, under the assumption of isotropy, a $k^{-5/3}$ omnidirectional wave number spectrum for (electron) density fluctuations in the local interstellar medium and (proton) density fluctuations in the solar wind [*Armstrong et al.*, 1981, and references therein; *Woo and Armstrong*, 1979; *Goldstein and Siscoe*, 1972; *Higdon*, 1984]. Generalizations of the incompressible Kolmogorov-Obukhov cascade theory to the case of variable density seem to lead, under the assumption that density fluctuations are expressible in terms of pressure fluctuations, to the prediction of a $k^{-7/3}$ omnidirectional ($k^{-13/3}$ modal) density fluctuation spectrum [*Batchelor*, 1951; *George et al.*, 1984, and references therein]. Theories to explain the discrepancy have been based on such plausible physical effects as anisotropy or passive convection [*Higdon*, 1984].

The purpose of this present note is to show that if the pressure fluctuation theory of *Batchelor* [1951] and *George et al.* [1984] is appropriately generalized to magnetohydrodynamics, and a density-pressure equation of state is assumed, a $k^{-5/3}$ omnidirectional density fluctuation spectrum emerges in a natural way, for wave numbers well above the reciprocal of the energy-containing eddy size. The calculation is straightforward and seems to require no additional assumptions beyond that of a $k^{-5/3}$ inertial-range spectrum for kinetic and magnetic energy.

The essence of the method is to iterate about Batchelor's expression for the incompressible pressure fluctuation spectrum, using an assumed equation of state (the precise form of which does not matter) to express the fractionally small density fluctuations. The equations of incompressible magnetohydrodynamics include

$$\partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla(p^*/\rho_0) + \mathbf{B}\cdot\nabla\mathbf{B}/4\pi\rho_0 + \nu\nabla^2\mathbf{v} \quad (1a)$$

$$\nabla\cdot\mathbf{v} = 0 = \nabla\cdot\mathbf{B} \quad (1b)$$

here $\mathbf{v}$ is the velocity field, $\mathbf{B}$ is the magnetic field, ρ_0 is the mass density (assumed uniform and constant), $p^* = p_m + B^2/8\pi$ is the total pressure, p_m is the mechanical pressure, and ν is the kinematic viscosity.

A Poisson equation for p^* results from taking the divergence of (1*a*) and using (1*b*):

$$\nabla^2(p^*/\rho_0) = -\nabla\cdot[\mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{B}\cdot\nabla\mathbf{B}/4\pi\rho_0] \quad (2)$$

If all fields are Fourier-decomposed over a large volume of the magnetofluid, the solution to (2) may be written in Fourier space as

$$p^*(\mathbf{k}) = (\mathbf{kk}/k^2):\sum_{\mathbf{q}}\{\mathbf{B}(\mathbf{k}-\mathbf{q})\mathbf{B}(\mathbf{q})/4\pi - \rho_0\mathbf{v}(\mathbf{k}-\mathbf{q})\mathbf{v}(\mathbf{q})\} \quad (3)$$

The mechanical pressure p_m is, also in Fourier space,

$$p_m(\mathbf{k}) = p^*(\mathbf{k}) - (8\pi)^{-1}\sum_{\mathbf{q}}\mathbf{B}(\mathbf{k}-\mathbf{q})\cdot\mathbf{B}(\mathbf{q}) \quad (4)$$

where $p^*(\mathbf{k})$ is given by (3).

Equations (3) and (4) have been derived neglecting compressibility. If the density fluctuations $\delta\rho$ are fractionally small, $\delta\rho/\rho_0 \ll 1$, we may assume that they may be related to p_m through an equation of state $p_m = p_m(\rho)$, giving $\delta\rho(\mathbf{k}) = p_m(\mathbf{k})/c_s^2$, where $c_s^2 \equiv dp_m/d\rho_0$ is the square of the sound speed, assumed infinite in incompressible flow. For a fluid in local thermodynamic equilibrium, the density fluctuations would have to be expressed in terms of both pressure and temperature fluctuations. The assumption that density is determined locally by pressure alone is common for a magnetofluid (which is usually far from thermal equilibrium in any case) and is correct in either the isothermal or the adiabatic limit. Using (3) and (4) then gives

$$\delta\rho(\mathbf{k}) = c_s^{-2}\Big\{(k_\alpha k_\beta/k^2)\sum_{\mathbf{q}}[B_\alpha(\mathbf{k}-\mathbf{q})B_\beta(\mathbf{q})/4\pi - \rho_0 v_\alpha(\mathbf{k}-\mathbf{q})v_\beta(\mathbf{q})] - (8\pi)^{-1}\sum_{\mathbf{q}}B_\alpha(\mathbf{k}-\mathbf{q})B_\alpha(\mathbf{q})\Big\} \quad (5)$$

where repeated Greek subscripts indicate vector components which are to be summed over.

The density spectrum may now be obtained by multiplying $\delta\rho(\mathbf{k})$ by its complex conjugate and ensemble averaging, which we indicate by angle brackets. The ensemble is the usual one for homogeneous turbulence theory [e.g., *Panchev*, 1971], and all members of it have identical mean density and total mass. We make the common assumption, a feature of most turbulence theories in some form or other, that we may approximate the resulting expectations of products of four Fourier coefficients by their quasi-normal [*Panchev*, 1971; *Whang*, 1977] values: $\langle 1234\rangle = \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle$, where the integers represent any individual Fourier coefficients of v_α or B_α. This is a familiar approximation, the limits of the utility of which have not been sharply delineated for

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hydrodynamic turbulence, and we shall not attempt their further justification for MHD here. In addition, we assume no net correlation between **v** and **B**, and further that the turbulence in both fields is homogeneous, isotropic, and nonhelical. The result for the density variance spectrum, after some tedious but straightforward algebra, is $c_s^4 S_\rho(\mathbf{k}) = F_{\mathrm{I}} + F_{\mathrm{II}} + F_{\mathrm{III}}$, where

$$F_{\mathrm{I}} = (8\pi^2)^{-1} \int d^3\mathbf{q}\, \{E_v(q)E_v(|\mathbf{k}-\mathbf{q}|)/|\mathbf{k}-\mathbf{q}|^4\} \sin^4\theta + (8\pi^2)^{-1} \int d^3\mathbf{q}\, \{E_B(q)E_B(|\mathbf{k}-\mathbf{q}|)/|\mathbf{k}-\mathbf{q}|^4\} \sin^4\theta \quad (6a)$$

$$F_{\mathrm{II}} = (8\pi^2)^{-1} \int d^3\mathbf{q}\, \{E_B(q)E_B(|\mathbf{k}-\mathbf{q}|)/|\mathbf{k}-\mathbf{q}|^4\} \cdot [(k\cos\theta - q)/q] \sin^2\theta \quad (6b)$$

$$F_{\mathrm{III}} = (32\pi^2)^{-1} \int d^3\mathbf{q}\, \{E_B(q)E_B(|\mathbf{k}-\mathbf{q}|)/|\mathbf{k}-\mathbf{q}|^4\} \cdot \{[k^2(1+\cos^2\theta) - 4kq\cos\theta + 2q^2]/q^2\} \quad (6c)$$

In (6) we have passed to a Fourier integral representation, appropriate to infinite volumes. $S_\rho(\mathbf{k})$ is proportional to $\langle|\delta\rho(\mathbf{k})|^2\rangle$, in the limit of infinite volume. The dimensions of $S_\rho(\mathbf{k})$ are, for example, M^2L^{-3}. $E_v(k)$ and $E_B(k)$ are the omnidirectional spectra of kinetic energy per unit volume and magnetic energy per unit volume, normalized so that

$$\int_0^\infty dk\, E_v(k) = \rho_0\langle v^2\rangle/2 \qquad \int_0^\infty dk\, E_B(k) = \langle B^2\rangle/8\pi$$

θ is the polar angle between **k** and **q**.

Equation (6*a*) without the magnetic term is *Batchelor*'s [1951] and *George et al.*'s [1984] result, if it is assumed that $p = p(\rho)$ only. The other terms are new. The extraction of inertial-range spectral laws comes down to the evaluation of the asymptotic behavior of the integrals in (6) for $k\lambda \gg 1$, where λ is the size of an energy-containing eddy. For purposes of these asymptotic evaluations (the main purpose here), we may work in the units for which $\lambda = 1$. A convenient energy spectrum for both E_B (which often seems to obey the Kolmogorov $-5/3$ law [*Matthaeus et al.*, 1982]) and E_v is one suggested by *von Karman* [1948] and employed by *George et al.* [1984]:

$$E(k) = Ck^4/(1+k^2)^{17/6} \quad (7)$$

where C is a constant independent of k. Only the small k and large k (Kolmogorov) behavior of (7) is used in the analysis, and the results seem to be independent of the details of the function near $k = \lambda^{-1} = 1$.

Except for k-independent multiplicative constants, F_{I}, F_{II}, and F_{III} can then be written, using (7), as

$$f_{\mathrm{I}} = k^{-13/3} \int_{-1}^{1} d(\cos\theta) \int_0^\infty d\eta\, (\eta^6 \sin^4\theta) \cdot [k^{-2}+\eta^2]^{-17/6}[k^{-2}+1+\eta^2-2\eta\cos\theta]^{-17/6} \quad (8a)$$

$$f_{\mathrm{II}} = k^{-13/3} \int_{-1}^{1} d(\cos\theta) \int_0^\infty d\eta\, \eta^6[\eta^{-1}\cos\theta - 1] \cdot \sin^2\theta[k^{-2}+\eta^2]^{-17/6}[k^{-2}+1+\eta^2-2\eta\cos\theta]^{-17/6} \quad (8b)$$

$$f_{\mathrm{III}} = k^{-13/3} \int_{-1}^{1} d(\cos\theta) \int_0^\infty d\eta\, \eta^6[\eta^{-2}(1+\cos^2\theta) - 4\eta^{-1}\cos\theta + 2][k^{-2}+\eta^2]^{-17/6} \cdot [k^{-2}+1+\eta^2-2\eta\cos\theta]^{-17/6} \quad (8c)$$

The asymptotic dependences of the three integrals in (8) are $O(1)$, $O(1)$, and $O(k^{2/3})$, respectively, implying that the three terms in $S_\rho(\mathbf{k})$ are $O(k^{-13/3})$, $O(k^{-13/3})$, and $O(k^{-11/3})$, respectively. These are modal density spectra and are converted into omnidirectional spectra by multiplication by $4\pi k^2$. The resulting $O(k^{-7/3})$ dependence of F_{I} and F_{II} is the same as Batchelor's result, as obtained by George et al. [*Batchelor*, 1951; *George et al.*, 1984]. The higher values of k will be dominated by F_{III}, which originates with variations in the magnetic pressure, which leads to the omnidirectional spectrum $k^{-5/3}$, and which seems to be close to the observations [*Armstrong et al.*, 1981; *Woo and Armstrong*, 1979; *Goldstein and Siscoe*, 1972]. It should be noted that the observations are made at wavelengths much greater than the Debye length, so that the difference between the electron density distribution and mass density distribution should be unimportant.

It will be noted that the density fluctuation given in (5) ignores any high-frequency "acoustic mode" contributions [e.g., *Kovasznay*, 1953]. It is, in effect, a quasi-static solution of the magnetohydrodynamic generalization of *Lighthill*'s [1952] equation, in the version of it appropriate to homogeneous turbulence. It is being assumed that "acoustic mode" contributions to $\delta\rho(\mathbf{k})$ are of higher order in the Mach number than (5); justifications of this assumption have been given for the aerodynamic case by *Lighthill* [1952, pp. 579–582].

Several obvious generalizations suggest themselves, such as the inclusion of correlations between **v** and **B** (cross helicity), and the inclusion of magnetic or mechanical ($\langle \mathbf{v}\cdot(\nabla\times\mathbf{v})\rangle$) helicity. External dc magnetic fields may be included only if they contain no spatial gradients. If the dc magnetic fields introduce no anisotropies or other departures from the Kolmogorov $k^{-5/3}$ energy spectra, they, too, give additional $k^{-5/3}$ density fluctuation spectral contributions. However, there is evidence that isotropy may be destroyed by the presence of a dc magnetic field [*Montgomery and Turner*, 1981; *Shebalin et al.*, 1983]. A lengthier treatment of these topics is planned.

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