Nearly incompressible magnetohydrodynamics at low Mach number

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The dynamics of a compressible magnetofluid plasma with a polytropic equation of state are considered in the limit of low plasma frame acoustic Mach number. The relationship between the equations describing the low Mach number flow and the equations of idealized incompressible magnetohydrodynamics is investigated using a multiple time scale asymptotic expansion procedure, which is justified by appealing to several rigorous theorems concerning both hydrodynamics and magnetohydrodynamics. When appropriate assumptions are adopted concerning the degree of departure from incompressibility, the lowest-order behavior is that of incompressible magnetohydrodynamics, associated with order Mach number-squared “pseudosound” density fluctuations. The first corrections to incompressible flow take the form of magnetosonic fluctuations, with associated pressure fluctuations at the same order as the pseudosound pressure. Resumming the asymptotic series gives rise to a simple set of equations that describes “nearly incompressible magnetohydrodynamics.” The theory provides a justification for the turbulent density spectrum theory of Montgomery, Brown, and Matthaeus [J. Geophys. Res. 92, 282 (1987)] and clarifies several issues pertaining to Alfven wave turbulence in the solar wind. The nearly incompressible description may also be useful in other theoretical contexts, particularly in extensions of incompressible magnetohydrodynamic turbulence theory, since it is expected to be valid for finite times (until possible shock structures form) when the global Mach number is sufficiently small.

I. INTRODUCTION

Studies of plasma dynamics in the magnetofluid approximation often involve a choice at the onset to describe the system entirely in the context of either incompressible magnetohydrodynamics (MHD) or compressible MHD. Turbulence theoretic studies, for example, have had a particular leaning toward use of the incompressible model. While there may be physical justifications for the incompressible turbulence model in some cases, this preference has largely been based on the benefits of relative analytical simplicity, practical considerations of finite computational resources, and the advantages of the similarity of incompressible MHD turbulence to its well-studied hydrodynamic counterpart.1,2 This approach has led to considerable progress in the theoretical understanding of incompressible MHD turbulence.3 There has also been some success in attempts to describe natural plasma turbulence, such as in the solar wind,4 in the context of incompressible theory.

However, there is rarely good reason to suppose that compressibility effects are totally absent in a magnetofluid. For example, density fluctuations are present in the solar wind plasma5 and have a measured wavenumber spectrum that is often consistent with the Kolmogorov $k^{-5/3}$ law.6 Similar density spectra are inferred for the interstellar medium7 using remote sensing techniques that are sensitive to electron density fluctuations. These observational results are particularly intriguing since they suggest some relationship to incompressible, homogeneous Kolmogorov theory, which has no obvious relevance to compressible media. Recently, Montgomery, Brown, and Matthaeus6 described a possible mechanism that may give rise to Kolmogorov-like density fluctuations in MHD. The essence of the model (which will be reviewed in further detail below) is that the magnetofluid is considered to be incompressible and the density fluctuations are perturbatively computed as a small amplitude response to the incompressible pressure spectrum. Under the assumptions that the magnetic field fluctuation spectrum is isotropic with a $k^{-5/3}$ wavenumber dependence and quasinormal statistics, Montgomery et al. found that the density fluctuation spectrum also has a $k^{-5/3}$ dependence at sufficiently high $k$. Notably, this model can account for the observed spectral law for density only when there are magnetic fluctuations present, and not when it is a pure hydrodynamic medium. The density fluctuations computed in the model may be termed “pseudosound,” in analogy with Lighthill’s8 hydrodynamic theory.

The MHD pseudosound prescription8 represents a promising direction in current attempts to understand and probe realistic compressible MHD turbulence, using the better understood model of incompressible MHD turbulence as a starting point. The purpose of the present paper is to develop a formalism that clearly describes the relationship between compressible and incompressible MHD turbulence in a certain limit, namely, that of low acoustic Mach number, in which the solutions of the two sets of equations may be regarded as close to one another in an appropriate sense. One result of this procedure will be that we may demonstrate explicitly the domain of relevance and the degree of accuracy of the pseudosound prescription for the density. The results

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will also be cast in terms of a more general "nearly incompressible" MHD model, which may provide a basis for further theoretical insights into the nature of compressible MHD turbulence.

The procedure, in outline, is as follows. First, the notation, units, and equations describing the incompressible and compressible MHD models are given in Sec. II, where simple physical arguments are also given for the structure of the relationship between the two models. In Sec. III, a multiple time scale analysis is set up to describe an asymptotic hierarchy of dynamical equations that describes low Mach number compressible MHD. Several orders of these equations are described in detail. In Sec. IV, we digress into a discussion of some important rigorous results of applied mathematics that have implications for the asymptotic analysis. In Sec. V, we construct the nearly incompressible MHD equations, using the leading-order terms of a slightly different asymptotic expansion. Some implications of the formalism are considered in Sec. VI, which includes further discussion of the pseudo-sound approximation, a description and resolution of a paradox of incompressible theory, and some commentary on possible further uses of the nearly incompressible model. The results are summarized and future directions are described in Sec. VII.

II. COMPRESSIBLE AND INCOMPRESSIBLE MODELS OF MHD

We consider the familiar equations of compressible, one-fluid MHD, which may be written in laboratory cgs units as an equation for the flow velocity \( u \),

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \frac{J \times B}{c},
\]

(1)

an MHD induction equation for the magnetic field \( B \),

\[
\frac{\partial B}{\partial t} = \nabla \times (u \times B),
\]

(2)

and a continuity equation for the mass density \( \rho \),

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u).
\]

(3)

In the above, \( t \) is the time, \( c \) is the speed of light, and \( p \) is the pressure, while the electric current density is \( J = (c/4\pi) \nabla \times B \). The magnetic field satisfies the initial constraint \( \nabla \cdot B = 0 \), which is preserved by subsequent evolution according to Eq. (2). In addition, we assume an equation of state in which the pressure depends only on the density (i.e., a polytropic relation of the form \( p = \rho^A \)), where \( A \) is a constant. The constant \( A \) may easily be evaluated in terms of a characteristic density \( \rho_0 \) and a characteristic sound speed \( c_s^2 = \partial p/\partial \rho |_{\rho_0} \), which we evaluate at density \( \rho_0 \). This puts the pressure in the form \( p = (\rho_0 c_s^2 / \gamma) (\rho / \rho_0)^\gamma \). For convenience we have omitted viscous dissipation terms from the right-hand side of Eq. (1) and Ohmic dissipation terms from the right-hand side of Eq. (2), though these effects are crucial in establishing turbulence effects through dissipative action at the very small scales. Since the dissipative terms will be considered to be linear, it would not be difficult to include them.

Having in mind that dissipation will be reinstated in the equations for purposes of application, the equation of motion (1) will be referred to below as the Navier–Stokes equation.

For the present purposes it is convenient to adopt dimensionless units in which \( t = t^* \), \( u = u^* U_0 \), \( \rho = \rho^* \rho_{0} \), \( p = p^* \rho_{0} \), \( B = B^* \rho_{0} U_0 \), and \( L = L^* L_{0} \), where \( L \) is a length, \( \nabla \) behaves as \( 1/L \), and \( t_0 = L_{0}/U_0 \). Then the equation of motion (1) may be written in dimensionless variables, after dropping the asterisks, as

\[
\rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \frac{1}{r_\lambda^2} (\nabla \times B) \times B.
\]

(4)

The dimensionless equation of state is

\[
p = \rho^*/\gamma M^2.
\]

(5)

The new quantities that appear in (4) are the acoustic Mach number, \( M = U_0 / c_s \), and the Alfvén number \( r_\lambda \) defined by \( r_\lambda^2 = U_0^2 / (B_0^2 / 4\pi \rho_0) \). For completeness, we note that \( \rho_0 = \rho_0 c_s^2 \gamma \), so that \( \gamma M^2 = \rho_0 U_0^2 / \rho_0 \). The quantity \( r_\lambda^2 / \gamma M^2 \) is often called the plasma beta. Instead of using the continuity equation, it will sometimes be convenient to replace it with an equation for the time evolution of the dimensionless pressure, namely,

\[
\frac{\partial p}{\partial t} + u \cdot \nabla p + \gamma p \nabla \cdot u = 0.
\]

(6)

The dimensionless induction and continuity equations have the same form as (2) and (3), so that Eqs. (2)–(4), or equivalently Eqs. (2), (4), and (6), along with the equation of state (5), will be subsequently referred to as the dimensionless compressible MHD equations.

The standard way to obtain the incompressible MHD equations from the compressible equations is to fix the density \( \rho = \rho_0 \), where \( \rho_0 \) is again a constant independent of position and the time. The continuity equation then becomes the time-independent constraint that \( \nabla \cdot u = 0 \). This constraint is nontrivial because it imposes a time-dependent condition on the pressure, which can be formally obtained from Eq. (4) by setting \( \rho = \rho_0 \) and requiring that the equation of motion be compatible with \( \nabla \cdot u = 0 \) for all time. In this case, computing the divergence of (1), one obtains a Poisson equation for the pressure, which in dimensionless units is

\[
\nabla^2 p = -\nabla^2 (B^2 / 2 r_\lambda^2) + \nabla \cdot \left[ (B \cdot \nabla B) / r_\lambda^2 - \rho_0 \nabla \cdot u \right],
\]

(7)

where a familiar identity has been used to expand the Lorentz force. With these ramifications in mind, Eqs. (2) and (4), together with the simple equation of state \( p = \rho_0 \) and the constraint \( \nabla \cdot u = 0 \), constitute the equations of incompressible MHD.

The well-known physically intuitive justification for the incompressible MHD model closely follows the similar argument for incompressible hydrodynamics in a homogeneous medium. Suppose that the sound speed is large compared to the speeds at which all other information propagates in the fluid; in MHD the relevant slow signals are due to convection and Alfvén wave propagation. Density irregularities will tend to relax by propagation of sound throughout the fluid. If convective motions and Alfvén wave propagation are sufficiently slow compared to the acoustic
relaxation time, those motions will occur in a medium that is always extremely close to the relaxed, constant density state. For consistency, the continuity equation implies that the velocity remains solenoidal. Consequently, for purposes of computing the evolution of the velocity field, one may assume the validity of the incompressible equation of motion.

This route to incompressibility clearly requires that \( c_r > U_0 \) and \( c_r > \nu_{\text{AO}} \), where \( \nu_{\text{AO}} = B_0/(4\pi \rho_0)^{1/2} = U_0/r_\lambda \) is the characteristic Alfvén speed. In the present work we will not discuss in any detail the possible existence of other limits that may also lead to incompressibility.10,11 However, such a limit, namely, \( \nu_{\text{AO}} \to \infty \), will be briefly discussed in a later section. The successes of incompressible models in astrophysics12 and dynamo theory13 as well as in numerous other applications in plasma and fluid dynamics strongly support the validity of the physical argument for the existence of this limiting form of the dynamics. However, the heuristic argument suffers from a lack of completeness, particularly with regard to the pressure. In the above argument for incompressibility, one simply drops the compressible equation of state in favor of constant density and the implied constraint represented by Eq. (7). Thus the role of the pressure appears to be entirely different in the two models. Ambiguities of this type occur in many physical systems with singular limits.14 In effect, the attainment of incompressibility imposes constraints on the simpler form of the dynamical equations that emerge in the low Mach number limit. In the following sections, particular attention will be given toward uncovering the nature of those constraints, and the conditions in which the limiting behavior smoothly approaches the incompressible MHD model.

An example of how this ambiguity of the role of the pressure can lead to difficulties is easily illustrated by the example of large amplitude Alfvén waves (e.g., Refs. 12 and 13). Consider the situation in which the magnetic field consists of a dc component \( b_0 \) and a fluctuation \( b \) of arbitrary amplitude, where the magnetic field is measured in Alfvén speed units. Suppose also that the density is uniform and the velocity field (in the same units) satisfies \( u = \pm b \) at some initial time. It is well known that in ideal incompressible MHD in an infinite homogeneous medium this initial condition admits a simple exact solution for all time, namely, that the large amplitude fluctuation propagates without distortion either along \( b_0 \) (for \( u = -b \)) or antiparallel to \( b_0 \) (for \( u = b \)). The same initial condition may be considered in the context of compressible MHD with a polytropic equation of state. In this case the propagating nondispersive solution exists only if, in addition to the above assumptions, the magnetic field amplitude \( |b_0 + b| \) is initially chosen to have a uniform value in space. The putative existence of such constant field amplitude states15 and their possible implications for the solar wind,16 have been actively discussed in the space physics literature. The usual arguments for incompressibility are of little help in understanding this discrepancy in the conditions necessary for these large amplitude waves to exist. This example will be further discussed in Sec. VI.

It is clear that a greater physical understanding of incompressible MHD and its relationship to compressible MHD requires an exploration of the limiting process that connects the two models at low Mach number. For the moment, the existence of incompressible MHD as the zero Mach number form of the MHD equations will be assumed. What we will attempt to uncover through a systematic formal procedure is the structure of the deviations from incompressibility, and especially the limiting form of the compressible pressure.

### III. ASYMPTOTIC ANALYSIS AT LOW MACH NUMBER

Our concern is with the limiting behavior of the compressible model when the sound speed greatly exceeds the characteristic flow speed, and while the Alfvén number \( r_\lambda \) is \( \mathcal{O}(1) \).

This ordering may be accounted for by defining \( \epsilon^2 \equiv \gamma M^2 \) and considering the effect of the condition \( \epsilon < 1 \) when Eq. (5) is substituted in Eq. (4). The main influence is that very large fluid accelerations are produced by small density variations. A Fourier component of a magnetofluid variable with wavenumber \( k \) will respond to the large pressure-induced accelerations with time variations of magnitude \( \partial / \partial t \sim \omega - k / \epsilon \) (in the adopted dimensionless units). These rapid rates of change are principally associated with sound wave propagation, as can easily be seen by the usual method of linearizing the dynamical equations about a uniform state. Meanwhile, the convective and Alfvén wave motions will have a characteristic rate \( \partial / \partial t \sim \omega - k \). As \( \epsilon \to 0 \) the two time scales become widely separated and the equations become very stiff. The low Mach number limit is a singular limit of the compressible equations, from which the incompressible MHD equations will emerge as a reduced description.

To account for the character of the functions \( \mathbf{B}(x,t) \), \( \mathbf{u}(x,t) \), \( \rho(x,t) \), and \( p(x,t) \) when \( \epsilon \) is small, each of them is assumed to be representable by a perturbation series. The misordering of pressure with respect to other terms in Eq. (4) is conveniently represented by defining \( \bar{p} \) according to \( p = \rho / \epsilon^2 = \bar{p} / \epsilon^2 \). Therefore we assume that

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \cdots, \\
\mathbf{B} &= \mathbf{B}_0 + \epsilon \mathbf{B}_1 + \epsilon^2 \mathbf{B}_2 + \cdots, \\
\rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots, \\
\bar{p} &= \bar{p}_0 + \epsilon \bar{p}_1 + \epsilon^2 \bar{p}_2 + \cdots.
\end{align*}
\]

(8)

The normalized pressure \( \bar{p} \) obeys the same equation as does \( p \), as can be seen by dividing Eq. (6) by \( \epsilon^2 \). Therefore, for the purposes of constructing the expansion, we will refer to the equation for \( \partial \bar{p} / \partial t \) as the pressure equation.

Next, since there are two widely separated time scales in the problem, they are represented by \( \zeta = t \) and \( \eta = t / \epsilon \). The slow variable \( \zeta \) represents convective and Alfvénic motions on the physical scale \( L_0 / U_0 \). Variations of the fast variable \( \eta \) with \( \mathcal{O}(1) \) magnitude represent physical motions on the scale of the characteristic sound crossing time \( L_0 / c_s \). Following the usual procedures of multiple time scale perturbation theory,17 the variables \( \zeta \) and \( \eta \) are treated as independent of one another. Time derivatives are written as

\[
\frac{\partial}{\partial t} = \frac{\partial \zeta}{\partial t} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = \frac{\partial \zeta}{\partial \zeta} + \epsilon^{-1} \frac{\partial \eta}{\partial \eta}.
\]
Second time derivatives become
\[
\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \xi^2} + 2 \varepsilon^{-1} \frac{\partial^2}{\partial \eta \partial \xi} + \varepsilon^{-2} \frac{\partial^2}{\partial \eta^2}.
\]

To formulate an asymptotic theory for small \( \varepsilon \), the series expansions in Eq. (8) are inserted into the compressible MHD equations, with the time dependence of all variables formally replaced by dependence of \( \eta \) and \( \xi \). We then equate terms that multiply like powers of \( \varepsilon \). In order to allow for the possibility of a uniformly valid asymptotic expansion, some additional constraints need to be imposed in certain circumstances. These arise because one would like to treat each higher-ordered term in expansions such as (8) as ever smaller corrections that lead to the full solution of the unexpanded problem, to the extent that this may be possible. For example, the expansion of a hypothetical variable \( y \) will be uniformly valid if \( \gamma_{j}/\gamma_{j-1} \) is bounded. Suppose that \( \gamma_{j} \) were to obey an equation of the form \( \gamma_{j}/\gamma_{j} = X(\xi) + Y(\xi, \eta) \) for reasonably well-behaved functions \( X \) and \( Y \). By formally integrating in \( \eta \), one sees that \( \gamma_{j} \) will increase linearly in the fast scale \( \eta \) since \( X \) is a constant with respect to \( \eta \) integration. In such cases it is useful to define an operator \( \langle \cdot \cdot \cdot \rangle \) that performs an average over the fast scale \( \eta \), and write \( \gamma_{j} = \langle \gamma_{j} \rangle + \gamma_{j}' \). Thus \( \langle \gamma_{j}' \rangle = 0 \). In addition, since all variables will be assumed to have finite variations in time, the averaging operator gives a zero result when operating on any quantity that may be written as a derivative with respect to \( \eta \). Consequently upon averaging, the model equations imply that \( \langle X(\xi) \rangle = X(\xi) = -\langle Y(\xi, \eta) \rangle \) and \( \gamma_{j}'/\gamma_{j} = Y - \langle Y \rangle \). The systematic application of this method eliminates the most troublesome pathologies in the expansion that would otherwise cause its failure. The new conditions that emerge are usually called solvability conditions.  We now give the results of applying these techniques to the first several orders of expansion of the compressible MHD equations. For convenience and simplicity of the development, periodic boundary conditions are invoked.

The leading-order term in the MHD equations is the \( O(\varepsilon^{-2}) \) pressure term in the Navier–Stokes equation, which is not balanced by any other term. This implies that \( \nabla \rho_{0} = 0 \), which is equivalent to the intuitive expectation that \( \rho_{0} \) is uniform in space. In \( O(\varepsilon^{-1}) \) the pressure equation gives \( \frac{\partial \rho_{0}}{\partial \eta} = 0 \) and the Navier–Stokes equation gives \( \rho_{0} \frac{\partial u_{0}}{\partial \eta} + \nabla \rho_{0} = 0 \), while the magnetic induction equation implies that \( \partial B_{0}/\partial \eta = 0 \).

Several important pieces of information are supplied by the \( O(\varepsilon^{0}) \) pressure equation,
\[
\frac{\partial \rho_{0}}{\partial \xi} + \frac{\partial \rho_{1}}{\partial \eta} + \gamma \rho_{0} \nabla \cdot u_{0} = 0.
\]

Since the first term does not depend on \( \eta \) and the second term is a derivative with respect to \( \eta \), the equation should be averaged over the fast time scale. The slowly varying part gives the relation
\[
\frac{\partial \rho_{0}}{\partial \xi} + \delta \rho_{0} \nabla \cdot \langle u_{0} \rangle = 0.
\]

Integrating over the volume and using the fact that \( \rho_{0} \) is spatially uniform and the assumption that no net mass enters or leaves the system, we conclude that \( \nabla \cdot \langle u_{0} \rangle = 0 \) and \( \partial \rho_{0}/\partial \eta = 0 \). Thus two important connections with incompressible theory already have emerged: the lowest-order density (and pressure) are constants in space and in both time scales, while the lowest-order flow velocity is solenoidal.

The fluctuating ingredient of the \( O(\varepsilon^{0}) \) pressure equation is also crucial. Since \( \langle \partial \rho_{1}/\partial \eta \rangle \equiv 0 \) and \( u_{0} - \langle u_{0} \rangle \equiv u_{0}' \), it reduces to
\[
\frac{\partial \rho_{1}}{\partial \eta} + \gamma \rho_{0} \nabla \cdot u_{0}' = 0.
\]

This may be combined with the \( O(\varepsilon^{-1}) \) Navier–Stokes result to find that
\[
\frac{\partial \rho_{1}}{\partial \eta} + \gamma \nabla \rho_{0} = 0.
\]

By taking the average of the \( O(\varepsilon^{-1}) \) Navier–Stokes equation, one also finds that \( \langle \rho_{1} \rangle = 0 \). Therefore \( \rho_{1} \) consists entirely of wave packets that vary on the fast time scale, propagate at the sound speed, and obey a sourceless wave equation. The involvement of the velocity field in this wave motion is only through the irrotational component of \( u_{0} \). If such waves are present and viscous dissipation is reinstated, they decay through the viscous damping of the associated velocity field. Moreover, if the waves are absent in the initial data, they cannot appear, since they have no sources. Notably, these waves are also associated with pressure \( (p) \) variations in the initial conditions that are unbounded as \( \varepsilon \rightarrow 0 \). In the following we assume that \( \rho_{1} \equiv 0 \) in the initial data, which is equivalent to the assumption of finite accelerations at vanishing Mach number. Thus, hereafter, as \( \varepsilon \rightarrow 0 \), \( \nabla \rho_{0} \) is finite and \( \nabla p_{0} \) is order \( \varepsilon^{2} \). This simplification proves to be essential in obtaining a well-behaved approach to incompressibility.

Incorporating these assumptions concerning the initial data, the results so far are summarized by
\[
\nabla \rho_{0} = \nabla p_{0} = 0, \quad \frac{\partial \rho_{0}}{\partial \eta} = \frac{\partial \rho_{0}}{\partial \xi} = 0, \quad \frac{\partial u_{0}}{\partial \eta} = 0, \quad \nabla \cdot u_{0} = 0.
\]

According to the relationship between pressure and density, we also have \( \rho_{0} = \rho_{0}, \quad \rho_{1} = \gamma \rho_{1} \equiv 0 \) (by assumption), \( \rho_{2} = \gamma \rho_{2}, \rho_{3} = \gamma \rho_{3}, \rho_{4} = \gamma (\gamma - 1) \rho_{4}^{2}/2, \) etc. The \( O(\varepsilon^{0}) \) Navier–Stokes equation is now written as
\[
\rho_{0} \left( \frac{\partial u_{0}}{\partial \xi} + u_{0} \cdot \nabla u_{0} \right) = -\nabla p_{2} + \gamma \rho_{0} \frac{\partial \rho_{1}}{\partial \eta} - \rho_{0} \frac{\partial u_{1}}{\partial \eta},
\]

where only \( \rho_{2} \) and the last term on the right-hand side admit dependence on the fast variable \( \eta \). Noting that \( \langle u_{0} \rangle = u_{0} \) and \( \langle \rho_{0} \rangle = \rho_{0} \) the average of (9) over the fast scale gives
\[
\rho_{0} \left( \frac{\partial u_{0}}{\partial \xi} + u_{0} \cdot \nabla u_{0} \right) = -\nabla p_{2} + \gamma \rho_{0} \frac{\partial \rho_{1}}{\partial \eta} - \rho_{0} \frac{\partial u_{1}}{\partial \eta},
\]

where the slowly varying pressure \( \bar{p}_{2} \equiv \langle \rho_{2} \rangle \) must obey the Poisson equation (7) in order to maintain the solenoidal property of \( u_{0} \). Equation (10) is the incompressible Navier–

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Stokes equation for MHD. The remaining (fast-varying) part of the \(O(\varepsilon^0)\) Navier–Stokes equation produces the relation
\[
\rho_0 \frac{\partial u_i'}{\partial \eta} + \nabla \tilde{p}_z' = 0.
\] (11)

The \(O(\varepsilon^1)\) magnetic induction equation may be averaged to give the induction equation for incompressible MHD, namely,
\[
\frac{\partial B_i}{\partial \xi} = \nabla \times \left( u_0 \times B_0 \right).
\] (12)

At this point the full set of incompressible MHD equations have been obtained, involving a solenoidal \(u_0\) obeying Eq. (10), a constant zeroth-order density, and a magnetic field obeying Eq. (12). Notably, the pressure obeys the appropriate Poisson equation (7) and is associated through the equation of state with density fluctuations that are \(O(\varepsilon^1)\), i.e., second order in the Mach number. These are the nonpropagating “pseudosound” density fluctuations discussed by Montgomery, Brown, and Matthaeus. They are determined by computing density variations \(\delta \rho \) according to
\[
\delta \rho = \varepsilon^2 \tilde{\rho}_z = \left( \varepsilon^2 / \gamma \right) \tilde{p}_z^*,
\] (13)
where the pressure term is computed from the Poisson equation (7), in accordance with the prescription suggested in Ref. 8. All of the quantities involved in the incompressible limiting form depend only on the slow time variable \(\xi\). However, it is important to keep in mind that the incompressible pressure \(\tilde{p}_z^*\) is not identical to the second-order pressure correction \(\tilde{p}_z\), but must be extracted from it by averaging over the fast time scale.

Corrections to the incompressible limit are also of interest. Subtracting (12) from the unaveraged \(O(\varepsilon^1)\) induction equation produces the restriction that \(\partial B_i / \partial \eta = 0\). Thus the first-order corrections to the incompressible magnetic field also vary only on the slow scale and cannot be involved in acoustic time scale fluctuations. Turning to the \(O(\varepsilon^1)\) pressure equation, we find that its average implies that \(\nabla \cdot \langle u_i \rangle = 0\) so that the purely slowly varying part of the first corrections to the solenoidal incompressible velocity is also a solenoidal field. The fast-varying part of the \(O(\varepsilon^1)\) pressure equation obeys
\[
\frac{\partial \tilde{p}_z'}{\partial \eta} + \gamma \tilde{\rho}_0 \nabla \cdot u_i' = 0.
\] (14)
Combining this with Eq. (11) again produces a sourceless wave equation,
\[
\frac{\partial \tilde{p}_z'}{\partial \eta} - \gamma \nabla^2 \tilde{p}_z' = 0,
\] \(\frac{\partial^2 u_i'}{\partial \eta^2} - \gamma \nabla^2 u_i' = 0.\) (15)
The wave motions implied by (15) propagate at the sound speed and have a structure similar to the one encountered previously for the fluctuations in \(\tilde{p}_z\). However, there are two important differences. First, the acceleration produced by \(\tilde{p}_z'\) is bounded as \(\varepsilon \to 0\), so there is no compelling reason to exclude them by assumptions about the initial data, as had been done previously for \(\tilde{p}_z\). Second, unlike the case encountered at first order, the magnetic field is involved in these fluctuations. This can be seen by examining the fast part of the \(O(\varepsilon^1)\) induction equation, as will be done presently. An important feature of the second-order acoustic time scale fluctuations in Eqs. (15) is that they contribute to the total physical pressure at the same order as the slowly varying pressure associated with the pseudosound density variations.

Next, the remaining \(O(\varepsilon^1)\) equations are considered. The fast-scale-averaged \(O(\varepsilon^1)\) Navier–Stokes equation is
\[
\rho_0 \left( \frac{\partial \langle u_i \rangle}{\partial \xi} + \langle u_i \rangle \cdot \nabla u_0 + u_0 \cdot \nabla \langle u_i \rangle \right) = - \nabla \langle \tilde{p}_z \rangle + \alpha^{-2} \{ (\nabla \times B_i) \times B_0 + (\nabla \times B_0) \times B_i \}.
\] (16)
In view of the previous determination that \(\nabla \cdot \langle u_i \rangle = 0\), we can see that \(\tilde{p}_z\) is determined by the Poisson equation resulting from computing the divergence of (16) in the same way that the ordinary incompressible pressure was determined at the preceding order. Thus the leading-order slowly varying correction to incompressible MHD flow is itself an incompressible flow. Similarly by consideration of the \(O(\varepsilon^1)\) induction equation, the next-order correction to the magnetic field, averaged over the fast time scale, obeys
\[
\frac{\partial B_i}{\partial \xi} = \nabla \times \left( u_i \times B_0 + u_0 \times B_i \right).
\] (17)
Consequently the fast-scaled-averaged \(O(\varepsilon^1)\) velocity, \(O(\varepsilon^1)\) pressure, and \(O(\varepsilon^1)\) magnetic field obey dynamical relations implied by (16), (17), and \(\nabla \cdot \langle u_i \rangle = 0\), which represent an incompressible MHD flow that is linearized about the leading-order incompressible solution. For the first time we see the suggestion that it may be possible to perform some type of resummation of the asymptotic series of equations that might render more compact the relationships between compressible and incompressible flow that are being described. This point will be reconsidered in Sec. V.

The fast scale \(O(\varepsilon^1)\) Navier–Stokes equation and induction equation are obtained by subtracting the averaged forms [Eqs. (16) and (17)] from the full equations (not shown). First, from the induction equation we find that
\[
\frac{\partial B_i}{\partial \eta} = \nabla \times (u_i \times B_0).
\] (18)
Referring to the pair of wave equations given in (15), it can be seen that \(B_i\) exhibits oscillations in response to the behavior of the fast second-order pressure and velocity. These waves act as magnetoacoustic waves in the limit that the sound speed is much larger than the Alfvén speed. Their velocity ingredient is purely compressional, since \(\nabla \cdot u_i = 0\) follows from (11). The velocity and pressure behave as ordinary sound waves and are not influenced by \(B_i\).

The behavior of the waves described by (15) and (18) has so far been determined only with regard to their oscillatory motion at the fast time scale \(\eta\). They also must respond to the slow scale convective motions, which are described by
the \(O(e^2)\) pressure equation and the \(O(e^1)\) Navier–Stokes equation upon subtraction of the fast-scale-averaged parts. The relevant results are

\[
\rho_0 \left( \frac{\partial u_i^*}{\partial t} + u_i^* \cdot \nabla u_0 + u_0 \cdot \nabla u_i^* \right) = -\nabla \tilde{p}_i^* - \rho_0 \frac{\partial u_i^*}{\partial \eta},
\]

\[
\frac{\partial \tilde{p}_i^*}{\partial \xi} + u_0 \cdot \nabla \tilde{p}_i^* = -\frac{\partial \tilde{p}_i^*}{\partial \eta} - \gamma \tilde{p}_0 \nabla \cdot u_i^*.
\] (19)

The solutions of these two equations are needed to determine the slow motions of \(u_i^*\) and \(\tilde{p}_i^*\) wave packets. However, the solutions are not immediately computable since the right-hand sides of both equations in (19) involve terms in \(u_i^*\) and \(\tilde{p}_i^*\) that are of higher order and have not yet been considered. This is in contrast to all other perturbative relations that have been encountered so far in this asymptotic expansion, which involved only "known" functions that had, in principle, already been solved for in lower order. It is not difficult to see that this complication does not go away in the next order. For example, the equation for \(\partial u_1^*/\partial \xi\) involves \(\tilde{p}_1^*\) and \(\partial \tilde{p}_1^*/\partial \eta\). This kind of mixing of differing orders of the expansion is not entirely limited to equations for the slow evolution of the rapidly varying wave packets. For example, the fast-scale-averaged \(O(e^2)\) pressure equation is

\[
\frac{\partial \tilde{p}_i^*}{\partial \xi} + u_0 \cdot \nabla \tilde{p}_i^* + \gamma \tilde{p}_0 \nabla \cdot (u_i^*) = 0.
\] (20)

Recall that \(\tilde{p}_i^*\) is the ordinary incompressible pressure and is already completely determined. Therefore (20) determines \(\nabla \cdot (u_i^*)\) and the fast-scale-averaged \(u_i\), in general will not be solenoidal. The compressional motions are induced by \(\tilde{p}_i^*\) and are direct consequences of the leading-order incompressible motion. Thus slow scale acoustic effects first enter the solution through the second-order correction to the velocity field. The clean separation between fast acoustic effects and slow incompressible convective effects obtained through the fast two orders breaks down at this order.

These problems do not imply a breakdown in the asymptotic relations between compressible and incompressible MHD that we have computed so far. Nor does it mean that higher-order closed form equations cannot be obtained. It may be possible, for example, to treat combinations of several higher-order equations simultaneously by appropriate resummation. Alternatively, there may be formulations of the asymptotic series of equations for compressible MHD that anticipate the complexities at higher orders and simplify their treatment. In Sec. V, a different approach will be suggested that preserves the results obtained here while also providing further insights into the structure of low Mach number MHD.

IV. RELATED RIGOROUS RESULTS

The asymptotic behavior of polytropic compressible MHD deduced in Sec. III depends upon application of a formal expansion procedure. The expansion assumes that the limiting behavior, which was found to be incompressible MHD, is approached in a suitably regular manner. Moreover, it was implicitly assumed that, if an MHD flow is initially close enough to incompressibility, it will, over time, remain close enough that the expansion remains valid. Some conditions for "closeness" were established in the previous section; \(M\) must be small and the density fluctuations must be \(O(M^2)\) both initially and thereafter. However, these conditions were either assumptions of the expansion procedure, or consequences of the elimination of secular terms in the expansion. Consequently the conclusions that we found, though physically appealing, are no more rigorous than the heuristic premises that motivated the procedure. Mathematical rigor is lacking. To firm up the asymptotic model one needs to elucidate the degree of regularity with which the incompressible limit is approached. The stability of the limiting process should be similarly clarified. In many applications of multiple scale asymptotic analysis the needed level of justification is established by providing explicit solutions to the limiting equations and the first few orders of the expansion. The nature of the limit and the uniformity of the perturbation series can then be explicitly demonstrated. In the present case this is not feasible, since we have in mind that the solutions are turbulent in nature and can generally be obtained only by numerical methods. In the absence of these solutions, the formal expansion procedure provided only limited information concerning the conditions for its own validity.

Recognizing that the analytical turbulence solutions are unavailable, another method to establish the nature of the \(M \to 0\) limit and its stability is to deduce rigorous properties of the solutions that permit these issues to be investigated. In recent years, applied mathematicians have actively studied these questions for the case of both ideal and viscous hydrodynamic flows with polytropic equations of state. A number of important rigorous results have been obtained in these studies of hydrodynamics. A few results are also available for MHD.\(^{18}\) The exact statement of the relevant theorems and their proofs, which involve lengthy functional analysis manipulations, would be well beyond the scope of this presentation. However, in view of the close relationship between hydrodynamic and MHD models, a summary of some of the hydrodynamic results is warranted.

The results paraphrased here are principally those of Klainerman and Majda, who studied the low Mach number relationship between polytropic compressible hydrodynamics and incompressible hydrodynamics.\(^{18,19}\) These correspond, respectively, to the set of equations (4)–(6) with \(B = 0\), and to (4), (7), and \(\nabla \cdot u = 0\). The main results for hydrodynamics are summarized in the important recent book by Majda.\(^ {20}\)\) but some of the explicit proofs are contained only in the original research papers.\(^ {18,19}\) The basic approach is to consider compressible flow and incompressible flow as initial value problems with closely related initial data. Initial data, the time-dependent solutions, various dynamical terms of the equations, and measures of the differences between relevant functions are characterized by norms in an appropriately chosen Sobolev space.\(^ {10,22}\)\) We forego any attempt to outline the proofs, but simply state the following heuristic summary of the relevant theorems.

The first theorem is that of uniform stability of the low Mach number limit. It is first established that for initial
flows that have small $O(M^2)$ density variations and bounded velocity gradients, a classical solution of the compressible hydrodynamic equations exists for a finite time $T$ and obeys additional essential inequalities. Using these results, the initial value problem is considered for which the initial velocity consists of a solenoidal flow plus an $O(M)$ deviation with bounded derivatives, and the density is a constant plus an $O(M^2)$ fluctuation with bounded spatial derivatives. For the latter problem it is shown that, as $M \to 0$, the compressible flow solution for the velocity converges to the solution of the incompressible equations with the same zeroth-order initial velocity and constant density. The convergence is weak with respect to a maximum norm and uniform in the space of functions that are locally continuous with continuous first derivatives. Convergence is obtained for the same time estimate $T$ obtained for the first problem. The incompressible limit is thus established to be stable, in the sense that compressible solutions that are initially near to incompressibility will remain nearby for finite times.

Second, it is shown that, for any finite time $T_p$, the Mach number may be chosen small enough so that $T > T_p$. Therefore the results of the first theorem can be applied for long times, provided that the initial data continue to satisfy the assumed ordering in $M$. This result is used to deduce the validity of linearized acoustics as a correction to the incompressible flow equations, contributing the pressure at the same order $[O(M^2)]$ as the incompressible pressure. Finally, it is shown that there exists a completely convergent expansion of the above described compressible flow problem about the incompressible limit in powers of the Mach number. The incompressible flow solution and the generalized linear acoustic equations are the leading-order terms. Higher-order corrections obey equations with structure similar to the acoustic equations.

Finally, Majda has shown that the breakdown of the asymptotic relationship described by the above theorems occurs as a result of the appearance of infinities in the maximum norm of either the time derivative of the flow velocity or its spatial derivatives. This is attributed to the formation of shocks.

Several important extensions to the above theorems have also been developed. Klainerman and Majda also considered the low Mach number limit of the compressible Navier–Stokes equation, obtained by adding a standard form for the viscous stress to the right-hand side of our Eq. (4) with $B = 0$, while retaining the polytropic form for the equation of state. In this approximation, effects of temperature on the pressure and the effects of heat flux are ignored. The theorems concerning the low $M$ limiting behavior that were obtained are essentially the same as for the ideal inviscid compressible case described above, though the method of proof was somewhat different.

The above cited mathematical results for the low Mach number limit of hydrodynamics provide obviously important justification for performing a formal expansion of compressible hydrodynamics near $M = 0$. In fact, such an expansion is easily obtained from the results in Sec. III by setting the magnetic field $B = 0$ everywhere. This would give, to two orders of expansion,

\[ u \approx u_0 + \epsilon u_1 + \cdots \]

and

\[ p \approx (1/\epsilon^2) [\rho_0 + \epsilon^2 (\bar{p}_2^* + \bar{p}_4^*) + \cdots] \]

for the form of the pressure and velocity, where, as before, $\rho_0$ is a constant uniform density, $u_0$ obeys the incompressible equations, $\bar{p}_2^*$ is the incompressible pressure, related to the pseudosound density fluctuations, and $u_1$ and $\bar{p}_4^*$ contain both fast and slow time variations. Though obtained by a formal elementary method, these terms agree in detail (cf. Sec. 2.4 of Ref. 20) with the rigorous asymptotic expansion developed by Klainerman and Majda. This agreement provides confidence that the related results for MHD including the magnetic field are correct, even though the mathematical theorems reviewed above pertain specifically to hydrodynamics.

The same kind of mathematical methods can be used to develop a rigorous theory for the low Mach number limit of compressible MHD. In fact, preliminary MHD results were presented in Ref. 18, where a slightly weaker form of the basic convergence theorem (relative to the above cited hydrodynamic case) was given for the $M \to 0$ bounded $r_\Lambda$ case. The pieces of the more general rigorous theory that appear to remain undone are the long-time stability theorem, the justification of acoustic wave corrections, and the general proof of the existence of a convergent series of corrections to the incompressible case, as well as the stronger form of the existence proof for the $M \to 0$ limit. A second type of compressible MHD limit has also been considered, in which $r_\Lambda \to 0$ while $M$ remains bounded. This is the strong magnetic field limit, which has been often invoked as leading to incompressibility in magnetofluids, especially in the context of the reduced MHD model that is often applied to tokamak studies. In contrast to the $M \to 0$ case, a weak convergence theorem was not obtained, in general, for the strong magnetic field limit. In particular, the limiting dynamical equation for the magnetic field does not reduce to its proper incompressible MHD form, unless the condition that $\nabla \times B = 0$ is supposed to be true everywhere in the fluid. This result suggests that the low Mach number route to MHD incompressibility is considerably more effective than the strong magnetic field limit.

It would clearly be desirable to extend the rigorous MHD results further to the level of completeness of the hydrodynamic results. Extensions to include viscous and resistive dissipation and heat flux would also be of great interest. Furthermore, the difficulties with the strong field limit to incompressibility may warrant further investigation. However, these are well beyond the scope of the present paper. For the present, we will be content that the cited rigorous results for hydrodynamics and MHD lend considerable credence to the conclusions that may be drawn from the asymptotic analysis of low Mach number MHD that were given in Sec. III.

V. NEARLY INCOMPRESSIBLE MHD MODEL

The sequence of equations developed in Sec. III unravels the dynamical structure of low Mach number compress-
ible MHD. We found that the first few orders of the expansion obey simple equations, and include simply interpreted effects—incompressible MHD flow [Eqs. (10) and (12)] accompanied by incompressible pressure variations, followed by slow incompressible corrections [Eq. (16)] and fast magnetoacoustic waves [Eqs. (15) and (18)]. This structure may provide some physical insight but falls short of providing a useful dynamical representation because the long time scale variations of the magnetoacoustic wave packets couple to fast variations at higher order in \( M \). However, the rigorous developments in the area of hydrodynamics discussed in Sec. IV suggest that a more compact representation of the lowest-order departures from incompressibility may also exist for MHD. An attempt at a statement of such a model is now given here.

Following the hydrodynamic\(^{19,20}\) and MHD\(^{18}\) results obtained by Klainerman and Majda, assume that the Mach number is low enough that for suitable initial data the compressible MHD equations have classical solutions satisfying the long-time existence properties, and that an asymptotic low Mach number expansion exists and can be shown to be completely convergent. The details of the mathematical sense of the convergence will not be of concern here. With these assumptions it is tempting to assume an expansion of the solutions of the compressible MHD equations that is similar to that given in Eq. (8). Let \( u = u_0 + \epsilon u' + \cdots, B = B_0 + \epsilon B' + \cdots, \) and \( \rho = \bar{\rho}_0 + \epsilon (\bar{\rho}^\infty + p') + \cdots \), where \( \rho = \bar{\rho}/\epsilon^2 \) and \( p \) are again connected by the polytropic relation. In the spirit of the above assumptions, the zeroth-order solutions obey the incompressible MHD equations with \( \rho_0 = \rho_0 \) a uniform constant, and \( \nabla \cdot u = 0 \). In contrast to the multiple time scale expansion used in Sec. III, here the fast and slow scales will be allowed to remain mixed in terms of higher order than the \( M = 0 \) limiting incompressible solutions.

Substituting the above expansion into the Navier–Stokes equation (4) and subtracting the \( O(\epsilon^2) \) terms that obey the incompressible MHD Navier–Stokes equation, we find

\[
\epsilon \rho_0 \left( \frac{\partial u'}{\partial t} + u_0 \cdot \nabla u' + u' \cdot \nabla u_0 \right) = - \frac{\nabla p'}{\epsilon} + \frac{\epsilon}{r^2} \left[ (\nabla \times B_0) \times B' \right] + O(\epsilon^2).
\]

(21)

After analogous manipulations with the induction equation one obtains

\[
\epsilon \frac{\partial B'}{\partial t} = \epsilon [\nabla \times (u' \times B_0 + u_0 \times B')] + O(\epsilon^2).
\]

(22)

In terms of the expanded variables, without any subtraction, the pressure equation becomes

\[
\epsilon^2 \left( \frac{\partial (\bar{\rho}^\infty + p')}{\partial t} + u_0 \cdot \nabla (\bar{\rho}^\infty + p') \right) = - \epsilon \gamma \bar{\rho}_0 \nabla \cdot u' + O(\epsilon^3).
\]

(23)

In standard perturbation analyses, all terms having the same order in \( \epsilon \) would be treated on the same footing, and a distinct equation is written for each order. Here, however, since the slow convective motion and the fast acoustic motion have not been separated, a mechanism is needed to allow the occurrence of the rapid accelerations that are characteristic of low Mach number magnetoacoustic waves. This can be done, as suggested by the form in which (21)–(23) have been written, by allowing the pressure and the divergence of the velocity field to appear in a superficially misordered way. No pathologies are introduced into the lower-order equations by this procedure, since we have assumed that the incompressible model is obtained at the lower orders, so that \( \bar{\rho}_0 \) is uniform and \( O(\epsilon^4) \) pressure fluctuations are absent. Consequently, the higher-order terms symbolically written in (21)–(23) are dropped, and the remaining terms are considered to be exact at this order. The discarded terms are considered to enter into higher-order equations, and the simplified forms of (21)–(23) are taken to be the first set in a series of equations for corrections to incompressibility. After some simple transformations, these become

\[
\rho_0 \left( \frac{\partial u'}{\partial t} + u_0 \cdot \nabla u' + u' \cdot \nabla u_0 \right) = - \frac{\nabla p'}{\epsilon} + \frac{1}{r^2} \left[ (\nabla \times B_0) \times B' + (\nabla \times B') \times B_0 \right],
\]

(24)

\[
\frac{\partial B'}{\partial t} = \nabla \times (u' \times B_0 + u_0 \times B'),
\]

(25)

and

\[
\frac{\partial p'}{\partial t} + u_0 \cdot \nabla p' + \frac{\gamma p}{\epsilon} \nabla \cdot u' = F,
\]

(26)

where \( F = - (\partial \bar{\rho}^\infty / \partial t + u_0 \cdot \nabla \bar{\rho}^\infty) \) is the negative of the convective derivative of the incompressible pressure.

Equations (24)–(26) represent the leading-order terms in an asymptotic expansion of compressible MHD at low Mach number. This expansion is not identical to the one developed in Sec. IV by multiple scale techniques. In fact the equations that emerged from the multiple scale method can be reassembled into a form very similar to the newer set, but at each order terms are left over that are not disposable. The new expansion affords a means of avoiding the problem of mixing of orders that was discussed in Sec. IV. This is accomplished at the expense of leaving a misordered pressure term in the equations for the corrections, which is argued to be responsible for magnetoacoustic time scale motions.

Equations (24)–(26) are also quite closely related to the first corrections to incompressibility in the asymptotic expansion developed for the hydrodynamic case by Klainerman and Majda.\(^{19,20}\) Thus it is reasonable to conjecture that these equations can provide the starting point for the proofs of the MHD versions of the rigorous theorems alluded to in Sec. IV.

Following the terminology used by Klainerman and Majda, Eqs. (24)–(26) may be called "the equations of generalized magnetoacoustics." These equations have several
features that deserve mention. Unlike the magnetoacoustic variables in the multiple time scale expansion, these equations for the compressible motions contain both slow and fast time scales, and are a closed set of dynamical equations. The misordered pressure and divergence terms cause these equations to be stiff, but not to the extent of the original compressible MHD equations, since only one power of \( \varepsilon \) appears in the denominator. The original flow equation (4) contains an \( \varepsilon^2 \) in the denominator of the pressure term through the equation of state (5). The coupling between the dominant incompressible solutions and the magnetoacoustic corrections occurs through "linearized" convective operators and also through the forcing function \( F \) that depends on the incompressible pressure and therefore is nonlinearly related to the incompressible flow velocity and magnetic field.

Combined with the incompressible MHD equations, which provide the time evolution of \( u \) and \( B \), the generalized magnetoacoustic equations give a prescription for the time evolution of the leading-order compressible corrections. At sufficiently low \( M \) and for properly ordered initial data, the solutions to the compressible MHD equations may be well approximated by the sum of the incompressible solutions and the solutions of the generalized magnetoacoustic equations. Prior to times at which shock formation might cause the expansions to break down, this model of "nearly incompressible MHD" may be useful in analytical or numerical treatments of compressible MHD problems.

VI. LOW MACH NUMBER DENSITY FLUCTUATIONS, PSEUDOSOUND, AND ALFVEN WAVES

A full understanding of the implications of the asymptotic low Mach number expansion of MHD and of the nearly incompressible MHD model will require considerable further study. The present remarks will be confined to some general physical effects that follow naturally from the low Mach number MHD theory presented above.

First, consider the question: How close can a compressible MHD medium be to incompressibility at a small but nonzero Mach number? For definiteness, assume that homogeneous MHD turbulence is the system under consideration, and that it may be modeled by periodic boundary conditions. The validity of the asymptotic analysis depends upon having initial data that satisfies the ordering

\[
\begin{align*}
&u \approx u_0 + \varepsilon u_1 + \cdots, \\
&B \approx B_0 + \varepsilon B_1 + \cdots, \\
&p \approx (1/\varepsilon^2) [ \rho_0 + \varepsilon^2 (\tilde{p}_r + \varepsilon') + \cdots],
\end{align*}
\]

(27)

where the total leading-order pressure correction is \( \tilde{p}_r = \tilde{p}_r + \varepsilon' \). These orderings in \( \varepsilon = \sqrt{M} \) are expected to remain valid until the higher-ordered terms in the asymptotic expansion become dynamically misordered due to the appearance of shocks or other discontinuities. The question of whether \( M \) can be chosen small enough so that this breakdown does not occur in finite time is a matter of considerable mathematical subtlety and has not been addressed here. The rigorous results discussed in Sec. IV suggest that there will always be at least some finite time for which the expansions remain properly ordered. During that period, \( u, B, \) and the constant \( \rho_0 \) are solutions of the incompressible MHD equations, with \( \tilde{p}_r \) being the incompressible pressure. The leading-order corrections, \( u_1, B_1, \) and \( p' \) contain convective, Alfvénic, and acoustic time scale phenomena. The time evolution of the leading-order corrections can be treated either by the multiple scale method of Sec. III, or by the more compact generalized magnetoacoustic model in Sec. V.

The answer to the question posed above is simple, but perhaps not obvious. To produce a closely incompressible evolution, one might be tempted initially to prepare the magneto-fluid with exactly uniform density and solenoidal velocity, so that \( \rho_2 = \tilde{p}_r/\gamma = 0 \) and \( u_1 = 0 \). According to Eqs. (24) and (27), this "perfectly incompressible" initial data includes magnetoacoustic activity from the onset, since \( p'(t = 0) \neq 0 \). Non-negligible acoustic time scale fluctuations in the pressure occur immediately, through the interaction of Eqs. (24) and (26). The magneto-fluid immediately departs from incompressibility at order \( M^2 \) in the pressure and in order \( M \) in the velocity and magnetic fields. Comparing this response to the incompressible terms, the pressure is seen to have a leading-order admixture of incompressible behavior. Alternatively, the magneto-fluid may be again prepared with solenoidal velocity, but with initial \( O(\varepsilon^2) \) density perturbations chosen so that \( p' = \tilde{p}_r = 0 \) initially. From Eq. (24) this implies that \( \partial u_1 / \partial t = 0 \) at \( t = 0 \), so there are no rapid accelerations of fluid elements initially. Consequently there will be no magnetoacoustic waves initially, and all fast scale time derivatives vanish at \( t = 0 \). Magnetoacoustic activity will, in general, appear in time due to the influence of the slowly varying forcing function \( F \). Recalling that \( \tilde{p}_r \) is the pressure associated with the incompressible component of the flow, it is clear that with this choice of initial data the leading-order departures from incompressibility are initially at least one order higher in \( M \).

Thus one is led to the somewhat surprising conclusion that the compressible MHD initial value problem that remains closest to incompressibility does not have perfectly uniform density. This type of conclusion appears to have first been pointed out by Kainerman and Majda for the hydrodynamic case. An interesting application of this idea may be to produce "quiet start" initial conditions for low Mach number compressible MHD simulations.

Another important consequence of the asymptotic form of the compressible equations is the natural way that it leads to the basis for the pseudosound prescription for calculating nearly incompressible MHD density fluctuations. The pseudosound hypothesis for MHD, which may be thought of as a quasistatic treatment of Lighthill’s density equation extended to MHD (e.g., Ref. 26), computes density fluctuations as a linear response to the incompressible pressure, according to Eq. (13) above. Two restrictions on the use of (13) have become evident in the present developments. First, the incompressible pressure represents a leading-order contribution to the dynamical pressure under the conditions that \( M \) is small and that \( O(\varepsilon^4) \) density fluctuations are absent in the initial data. In general, the second-order pressure also includes contributions from the magnetoacoustic terms,
and the full leading-order density fluctuation is \( \delta \rho = e^{\gamma-1}(\rho^2 + \rho') \). The pseudosound prescription ignores the additional contribution \( \rho' \). The additional term includes both fast and slow scales, although, according to Eq. (14), it vanishes when averaged over the fast acoustic time scale. [Note that for the purpose of this argument there are no important differences between the \( O(\epsilon^2) \) pressure corrections \( \rho^2 \) in the multiple scale analysis, and \( \rho' \) in the generalized magnetoacoustic expansion.] Consequently, the accuracy of the pseudosound approximation at low Mach number critically relies not only on the absence of misordered \( O(\epsilon^4) \) density variations but also on the suppression of magnetoacoustic density variations that enter at the same order as the pseudosound. The latter may be physically suppressed by the selection of initial data, or by strong dissipation of compressive fluctuations, which may occur through MHD mechanisms or by kinetic processes such as Landau damping (e.g., Ref. 5). In practical data analysis situations, the influence of the magnetoacoustic fluctuations might be suppressed by low-pass filtering, leaving only the slowly varying pseudosound density contributions in the leading order.

Another ramification of the ideas presented here is the simple explanation of the paradoxical distinction between large amplitude incompressible Alfvén waves and their low Mach number compressible counterparts, which was described at the end of Sec. II. The \( \rho = \rho_0 = \) uniform constant incompressible solutions \( u = \pm b \) are now interpreted as the correct leading-order solutions to the asymptotic hierarchy of compressible equations. Accompanying the large amplitude Alfvén waves in the compressible case are pseudosound density fluctuations that must be present whenever the Poisson equation (7) has a nontrivial solution. This will occur for the Alfvén wave states whenever \( \nabla B \) is nonzero. Therefore spatial variations of the magnetic field magnitude are balanced by \( O(M^2) \) density fluctuations at low Mach number, and the search for strictly constant \( |B| \) Alfvén wave solutions is unnecessary. The asymptotic theory indicates that the incompressible Alfvén wave solutions are valid leading-order compressible solutions at low Mach number even when magnitude variations of the \( O(\epsilon^0) \) magnetic field are fractionally large.

VII. DISCUSSION AND CONCLUSIONS

In this paper the relationship between the incompressible MHD model and the compressible MHD model with polytropic equation of state has been investigated in the limit of small acoustic Mach number. Two closely related asymptotic expansions were developed to the first few orders of Mach number. In the first, a multiple time scale analysis was used, which provides several orders of cleanly separated equations for the first few orders [Eqs. (10)–(20)]. The second method, motivated by rigorous mathematical results mainly proved for low Mach number hydrodynamics by Klainerman and Majda,16–20 gives rise to a set of closed dynamical equations for the lowest-order departures from incompressibility. These equations of nearly incompressible MHD [Eqs. (24)–(26)] consist of the incompressible MHD equations and a set of generalized magnetoacoustic equations. It is suggested, again following the hydrodynamic results obtained by Klainerman and Majda, that the nearly incompressible MHD model is a close approximation to the behavior of compressible MHD for a finite time.

The two asymptotic methods are distinct, but there is a substantial overlap in the information provided by the two expansions. Both asymptotic methods include the same set of restrictions on the initial data and the same ordering and physical interpretation of the first few orders of the solutions, notably including incompressible MHD at \( O(1) \). Both cases exclude the possibility of \( O(M) \) density fluctuations in the initial data. Other restrictions on the initial data probably also are needed. In analogy with the requirements found in rigorous mathematical convergence proofs in hydrodynamics,20 one probably also needs to have suitably bounded spatial derivatives of the velocity, density, and magnetic field in the initial data. These precise conditions have not been investigated here.

The behavior of the pressure was a particular focus of attention. The leading-order fluctuations in the pressure include the incompressible pressure, determined in the usual way from a Poisson equation and involving only slow time scale variations, and a magnetoacoustic contribution that always includes fast time dependence. Notably, the incompressible pressure is related to the density by the MHD pseudosound density theory introduced by Montgomery et al.8 Subject to possible complications arising from the presence of the magnetoacoustic waves at the same order in Mach number, the present theory provides a justification of the pseudosound description. The clarification of the relationship between pressure and density in the compressible and incompressible MHD models also provides insight into the low Mach number structure of large amplitude Alfvén waves.

There appears to be considerable potential for the use of the above asymptotic expansions in investigations of the structure of low Mach number compressible MHD. In particular, the incompressible model [Eqs. (10), (12), and \( \nabla \cdot u = 0 \)], together with the generalized magnetoacoustic equations, have the advantage of being a closed dynamical system. This nearly incompressible MHD model may prove to be useful for numerical simulations in certain circumstances. The nearly incompressible model may also provide a basis for extension of some of the ideas of incompressible MHD turbulence theory2,3,27,28 to treat the case of low Mach number compressible MHD turbulence. An immediate consequence of these developments lies in the extension of MHD theory of solar wind fluctuations5,6,29 to treat local compressibility related effects. In that context, the pseudosound theory has already provided a basis for interpretation of Kolmogorov-like density spectra in the interplanetary and interstellar plasmas. A more complete treatment of interplanetary turbulence in the present perspective is planned. Finally, it is clear that future developments in the area of applied mathematics may provide a more rigorous basis for the physically motivated approach presented here.

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9Numerical verification of the accuracy of the pseudosound approximation of two-dimensional isothermal MHD have been presented by J. Shebalin and D. Montgomery [J. Plasma Phys. 39, 339 (1988)].