The idea of critical balance in Kolmogorov turbulence

As I mentioned in my notes on the Kolmogorov 1941 scaling argument for the omni-directional turbulent wavenumber spectrum (and the PSST and PoP review articles), the key idea is that magnetic energy is transferred at a rate b^2/τ_b, where b is the fluctuating part of the magnetic field, without dissipation from a larger scale to a smaller scale.

The kinetic energy (per mass) is transferred at a rate: $\epsilon \sim v^2/\tau_v$. We can write

$$\tau_v = \frac{\ell}{v}$$

where v is the typical velocity fluctuation at the scale ℓ. So,

$$\epsilon \sim \frac{v^3}{\ell} \sim v^3 k$$

or $v \sim (\epsilon \ell)^{1/3}$. This contains all the information we need to show the $-5/3$ cascade. It comes from

$$E_v(k, \epsilon) = C k^{\alpha} \epsilon^{\beta}$$

and $E_v(k) \propto v^2/k$.

We found that since

$$\int E_b(k) dk = \langle b^2 \rangle$$

and $E_b(k) \propto b^2/k$ we could assume that for MHD, that the dynamical time was an Alfvén crossing time at the scale ℓ:

$$\tau_{MHD} = \frac{\ell}{v_A} \sim \frac{1}{kb}.$$

This is because $\omega_{MHD} = kv_A$. This gave us the same $-5/3$ scaling as we found for $E_v(k)$.

Now, consider magnetic turbulence in a medium with a strong, well-defined background field B_0. This is surely the case for SSX, and even though β in the solar wind in close to unity, and the fluctuation level can be 100%, it is often the case in the solar wind as well. In such a case, it is not obvious that fluctuations along and across the background magnetic field should have the same character. Indeed, it is often observed in space and laboratory plasmas that there is an anisotropy between fluctuations in $k_{\parallel}$ and $k_{\perp}$.

The theory to explain the anisotropy started with Montgomery and his collaborators in the 1980s, and was refined by Goldreich and Sridhar in the

1990s (see also the review article by Schekochihin in 2009). They made the important assumption that the timescales associated with energy transfer along and across the background magnetic field should be the same:

$$\omega \sim k_{\|} v_{Alf} \sim k_{\perp} v_{\perp}$$

here $v_{\perp}$ is the fluctuating velocity from above, and is much slower than the Alfven speed v_{Alf}. This also implies in the assumption that the scale along the magnetic field is longer than the scale across: $L_{\|} \gg \ell_{\perp}$.

So there is a single turbulent transfer time:

$$\tau \sim \frac{L_{\|}}{v_{Alf}} \sim \frac{\ell_{\perp}}{v_{\perp}}$$

In the perpendicular direction, we recover the Kolmogorov result from above for the kinetic energy in the perpendicular direction:

$$\epsilon \sim \frac{v_{\perp}^3}{\ell} \sim v_{\perp}^3 k_{\perp}$$

Since, for frozen in flow, we have $b_{\perp} = v_{\perp}\sqrt{\mu_0 \rho}$, so we also have the $-5/3$ result for $E_b(k_{\perp})$. It turns out that the prediction for $E_v(k_{\|}) \sim k^{-2}$. Note that the turbulence is three dimensional, with correlations parallel and perpendicular to the background magnetic field related at each scale by the critical balance assumption.