Some notes on MHD equations of state (Chew Goldberger Low)

The simplest MHD equation of state treats the plasma like an ideal gas with pressure: $P = nkT$ (n is the number of particles N per volume). This model just has one fluid and one temperature. The more sophisticated model, appropriate for magnetized plasma with long mean free paths ($\omega_{ci}\tau_{coll} > 1$) is due to Chew, Goldberger, and Low (1956).

Simple MHD model: If there's just one pressure, and we assume an ideal gas, then $P = nkT$ and for adiabatic processes, PV^γ is a constant. An alternate way to write the adiabatic equation of state is:

$$\frac{d}{dt}\left(\frac{P}{n^\gamma}\right) = 0$$

This is because the relationship

$$\frac{P}{n^\gamma} = \frac{nkT}{n^\gamma} = const$$

implies $n^{1-\gamma}T = const$. This is the same as $PV^\gamma = const$ since

$$PV^\gamma = PVV^{\gamma-1} = NkTV^{\gamma-1} = const$$

So $TV^{\gamma-1} = const$ and $V = N/n$. This is a simple model, but ignores the effect of the magnetic field, and if the mean free path is long (ie if $\omega_{ci}\tau_{coll} > 1$) then the model doesn't take into account that pressure along the magnetic field can be different than pressure across.

Note that here γ is the standard ratio of specific heats. It can be shown in statistical mechanics that

$$\gamma = \frac{f+2}{f}$$

where f is the number of degrees of freedom. f is typically 3 (for a 3D process) in which case $\gamma = 5/3$, but f could be as small as 1 (for 1D compression) in which case $\gamma = 3$. If $TV^{\gamma-1} = const$ and $\gamma = 3$, then an adiabatic compression of a factor of 10 implies a temperature increase of a factor of 100.

CGL equation of state: We saw above that the simple MHD adiabatic equation of state is:

$$\frac{d}{dt}\left(\frac{P}{n^\gamma}\right) = 0$$

The CGL or double adiabatic theory has two formulae:

$$\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0$$

$$\frac{d}{dt}\left(\frac{P_\parallel B^2}{n^3}\right) = 0$$

These come from adiabatic constants for single particle motion in a magnetic field. I may review that below.

The first adiabatic invariant is the magnetic moment $\mu = IA$. It turns out that the magnetic moment can be written:

$$\mu = \frac{W_\perp}{B} = const$$

where $W_\perp$ means the part of the proton's kinetic energy associated with motion perpendicular to the magnetic field. Since $P_\perp = nW_\perp$ (pressure is always an energy per volume), we have right away the first of the CGL equations:

$$\left(\frac{P_\perp}{nB}\right) = const$$

The second adiabatic invariant has to do with the parallel motion of a proton bouncing between two regions of strong magnetic field (called magnetic mirror in plasma physics) separated by a distance L: $v_\parallel L = const$. Now a little algebra.

First, there are a few other constants associated with a magnetic structure or flux tube. One is the total number of particles N = nV = nLA, where A is the cross-sectional area of the flux tube (it cancels out in a minute). The other constant is the magnetic flux $\Phi = BA$. Notice that from the first expression: A = N/nL. So, let's square the flux, then substitute in for the area:

$$\Phi^2 = B^2A^2 = \frac{B^2N^2}{n^2L^2}$$

Now we notice that since $P_\parallel = nW_\parallel$ and $W_\parallel \sim v_\parallel^2$, if we use the second adiabatic invariant: $v_\parallel L = const$, we find that $W_\parallel \sim 1/L^2$ and $P_\parallel \sim n/L^2$. So our flux equation becomes:

$$\Phi^2 = \frac{B^2N^2}{n^2L^2} \sim \frac{B^2}{n^3}\frac{n}{L^2} \sim \left(\frac{P_\parallel B^2}{n^3}\right) = const$$